Tempered solutions of $\mathcal{D}$-modules on complex curves and formal invariants

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Abstract

Let X be a complex analytic curve. In this paper we prove that the subanalytic sheaf of tempered holomorphic solutions of $\mathcal{D}_X$-modules induces a fully faithful functor on a subcategory of germs of formal holonomic $\mathcal{D}_X$-modules. Further, given a germ $\mathcal{M}$ of holonomic $\mathcal{D}_X$-module, we obtain some results linking the subanalytic sheaf of tempered solutions of $\mathcal{M}$ and the classical formal and analytic invariants of $\mathcal{M}$.

Contents

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Introduction

The search for algebraic or topological invariants of complex linear partial differential equations is classical and widely developed.

At the very first step of the study of linear differential equations, two main types of equations are distinguished: regular and irregular. To give an idea of the difference between the two kinds of equations, let us recall that, in dimension 1, the solutions of the former equations have moderate growth while the solutions of the latter have exponential-type growth.

The more general algebraic approach to the study of linear differential equations consists in considering differential equations as sheaves of modules over the ring $\mathcal{D}_X$ of linear differential operators on a manifold X. In this framework, in [K79] and [K84], M. Kashiwara gives a proof of the Riemann-Hilbert correspondence which is a generalization of the 21st Hilbert's problem. For X a complex analytic manifold, M. Kashiwara defines the functor $T\mathcal{H}om$ and he gives an explicit inverse to the functor of holomorphic solutions from the bounded derived category of complexes of $\mathcal{D}_X$-modules with regular holonomic cohomology to the bounded derived category of complexes of sheaves with constructible cohomology. This implies the classic result that the functor of holomorphic solutions $\mathscr{S}(\cdot)$ is an equivalence between the category of regular connections on X with poles along a closed submanifold Z and the category of linear representations of finite dimension of the fundamental group of $X \setminus Z$.

The irregular case is more complicated. In complex dimension 1, the classification of meromorphic connections through the formal classification and the Stokes phenomena is nowadays well understood. Let us roughly explain the formal classification. It is based on the Levelt-Turrittin's Formal Theorem. Such theorem is one of the cornerstones of the study of ordinary differential equations. For $\varphi \in z^{-1}\mathbb{C}[z^{-1}]$, set $\mathcal{L}^\varphi := \mathcal{D}_\mathbb{C} \exp(\varphi)$. The $\mathcal{D}_\mathbb{C}$-module $\mathcal{L}^\varphi$ is one of the basic examples of an irregular meromorphic connection. Given a finite set $\Sigma \subset z^{-1}\mathbb{C}[z^{-1}]$, a family of regular meromorphic connections indexed by Σ, $\{\mathcal{R}_\varphi\}_{\varphi\in\Sigma}$, one calls $\bigoplus_{\varphi\in\Sigma} \mathcal{L}^\varphi \otimes \mathcal{R}_\varphi$ a good model. Let $\mu_l : \mathbb{C} \to \mathbb{C}$, $z \mapsto z^l$ and μ_l^* the pull-back functor for germs of $\mathcal{D}$-modules. The Levelt-Turrittin's Formal Theorem states that, given a meromorphic connection $\mathcal{M}$, there exists $l \in \mathbb{Z}_{>0}$ such that

$$\mu_l^* \mathcal{M} \simeq \bigoplus_{\varphi\in\Sigma} \mathcal{L}^\varphi \otimes \mathcal{R}_\varphi \,, \tag{0.1}$$

as modules over the ring of linear differential operators with formal Laurent series coefficients. The formal isomorphism given in (0.1) does not induce an isomorphism in the analytic category, that is to say we can have two non-isomorphic meromorphic connections which are formally isomorphic. Such peculiarity is at the base of the Stokes phenomena. Roughly speaking, the

Hukuhara-Turrittin's Asymptotic Theorem states that the isomorphism (0.1) is analytic on sufficiently small open sectors. Indeed, such a theorem states that the holomorphic solutions of a meromorphic connection $\mathcal{M}$ on a small open sector S are $\mathbb{C}$-linear combinations of a finite number of functions of the form $h(z)\exp(\varphi(z^{1/l}))$, for $z^{1/l}$ an arbitrary l-th root of z defined on S, $\varphi \in z^{-1}\mathbb{C}[z^{-1}]$ not depending on S and h an invertible tempered holomorphic function on S with tempered inverse. The functions $\varphi(z^{1/l})$, which do not depend on S, are called determinant polynomials of $\mathcal{M}$.

In this paper we will use the description of meromorphic connections given by Ω-filtrations, introduced by P. Deligne in [DMR]. Let us briefly recall the setting of Ω-filtrations. We refer to [DMR], [Ma83], [BV] and [Ma91]. First one defines a local system Ω on S^1 whose stalks are partially ordered sets. Such local system represents the space of all possible determinant polynomials. Further, one defines the category of Ω-filtered and Ω-graded local systems. Roughly speaking, Ω-filtered local systems are obtained by gluing Ω-graded local systems preserving the local order of Ω. Now, the holomorphic solutions of a meromorphic connection can be endowed with a structure of Ω-filtered local system. Hence one gets a functor, $\mathscr{S}^{\Omega}$, which is an equivalence of categories between meromorphic connections (resp. formal meromorphic connections) and Ω-filtered (resp. Ω-graded) local systems. Roughly speaking, given a meromorphic connection $\mathcal{M}$ and $\vartheta \in S^1$, the Ω-graduation on $\mathscr{S}^{\Omega}(\mathcal{M})_{\vartheta}$ is determined by the positive integer l, the set Σ and the rank of the local systems $\mathscr{S}(\mathcal{R}_{\varphi})$ appearing in (0.1). The gluing morphisms defining the Ω-filtration on $\mathscr{S}^{\Omega}(\mathcal{M})$ are determined by the Stokes phenomena and the formal monodromy (i.e. the monodromy of the local systems $\mathscr{S}(\mathcal{R}_{\varphi})$). Even though the Ω-filtrations provide a complete description of the category of meromorphic connections, the local system Ω remains an analytic object built ad hoc. It is interesting to look for a topological description of the determinant polynomials.

Let us also recall that many sheaves of function spaces have been used in the study of irregular ordinary differential equations. For example, one can find in [Ma91] the definitions of the sheaves $\mathcal{A}^{\leq r}$ ($r \in \mathbb{R}$) defined on the real blow-up of the complex plane at the origin. In [DMR], P. Deligne defined the sheaves $\widetilde{\mathcal{F}}^k$, successively studied in detail in [LRP]. Roughly speaking, the solutions of $\mathcal{L}^{\varphi}$ with values in $\mathcal{A}^{\leq r}$ (resp. $\widetilde{\mathcal{F}}^k$) depend only on the degree and the argument of the leading coefficient of φ (resp. the degree and the leading coefficient of φ).

In higher dimension the study of irregular $\mathcal{D}$-modules is much more complicated. In [Sa00], C. Sabbah defines the notion of good model in dimension 2, he conjectures the analogue of the formal Levelt-Turrittin's Theorem and he proves it for meromorphic connections of rank ≤ 5. Recently, T. Mochizuki announced the proof of Sabbah's conjecture for the algebraic case.

Given a complex analytic manifold X, in [KS01], M. Kashiwara and P.

Schapira defined the complex of sheaves of tempered holomorphic functions $\mathcal{O}^t_{X_{sa}}$. The entries of the complex $\mathcal{O}^t_{X_{sa}}$ are not sheaves on a topological space but on the subanalytic site, X_{sa}. The open sets of X_{sa} are the subanalytic open subsets of X, the coverings are locally finite coverings. If X has dimension 1, then $\mathcal{O}^t_{X_{sa}}$ is a sheaf on X_{sa} and, for U a relatively compact subanalytic open subset of X, the sections of $\mathcal{O}^t_{X_{sa}}(U)$ are the holomorphic functions on U which extend as distributions on X or, equivalently, which have moderate growth at the boundary of U.

Further in an example in [KS03], M. Kashiwara and P. Schapira explicited the sheaf of tempered holomorphic solutions of $\mathcal{L}^{1/z}$. Such example suggests that tempered holomorphic functions and the subanalytic site could be useful tools in the study of ordinary differential equations. Roughly speaking, Deligne's idea of Ω-filtrations aimed to enrich the structure of the category of sheaves of $\mathbb{C}$-vector spaces where the functor of holomorphic solutions takes values. The approach through subanalytic sheaves enriches the topology of the space where the sheaf of solutions lives.

In this paper we go into the study of the subanalytic sheaf of tempered holomorphic solutions of germs of $\mathcal{D}$-modules. Denote by $\mathscr{S}^t(\mathcal{M})$ the subanalytic sheaf of tempered holomorphic solutions of a holonomic $\mathcal{D}_X$-module $\mathcal{M}$. Let $X \subset \mathbb{C}$ be an open neighborhood of 0, $\mathrm{Mod}(\mathbb{C}_{X_{sa}})$ the category of sheaves of $\mathbb{C}$-vector spaces on X_{sa}, GM_k be the subcategory of germs at 0 of $\mathcal{D}$-modules consisting of good models with Katz invariant $< k$. We prove that

$$\mathscr{S}^t\big(\,\cdot \otimes \mathcal{L}^{1/z^k}\big) : \mathsf{GM}_k \longrightarrow \mathrm{Mod}(\mathbb{C}_{X_{sa}})$$

is a fully faithful functor (**Theorem 3.1.5**). Further we prove that, given a germ of holonomic $\mathcal{D}$-module $\mathcal{M}$ with Katz invariant $< k$, the datum of $\mathscr{S}^t\big(\mathcal{M} \otimes \mathcal{L}^{1/z^{k+1}}\big)$ is equivalent to the data of the stalk of $\mathscr{S}^\Omega(\mathcal{M})$ and the local system underlying $\mathscr{S}^\Omega(\mathcal{M})$ (i.e. the local system of holomorphic solutions) (**Theorem 3.2.2**).

In conclusion we can say that tempered solutions on the subanalytic site give a topological description of the determinant polynomials of a given meromorphic connection. As further developement, it would be interesting to describe precisely the image category of the functor of tempered solutions in order to give a topological description of the space of determinant polynomials. It would also be interesting to give a good notion of Fourier trasform for tempered holomorphic solutions of algebraic $\mathcal{D}$-modules in the same spirit of [Ma91].

The present paper is subdivided in three sections organized as follows.

Section 1 is devoted to the definitions, the notations and the presentation of the main results that will be needed in the rest of the paper. In particular we recall classical results on the subanalytic sets and site, the tempered holomorphic functions and the germs of $\mathcal{D}$-modules.

The functions of the form $\exp(\varphi)$, $\varphi \in z^{-1}\mathbb{C}[z^{-1}]$, are the responsible for

the non-tempered-growth of the solutions of an irregular $\mathcal{D}$-module. This motivates the study of $\exp(\varphi)$ that we develop in **Section 2**. In particular, given $\varphi \in z^{-1}\mathbb{C}[z^{-1}]$ and U a relatively compact subanalytic open subset of $\mathbb{C}$, we give a necessary and sufficient topological condition on U so that $\exp(\varphi) \in \mathcal{O}^t_{\mathbb{C}_{sa}}(U)$. Further, we prove that the condition "for any $U \subset \mathbb{C}$ relatively compact subanalytic open set, $\exp(\varphi_1) \in \mathcal{O}^t_{\mathbb{C}_{sa}}(U)$ if and only if $\exp(\varphi_2) \in \mathcal{O}^t_{\mathbb{C}_{sa}}(U)$" is equivalent to "$\varphi_1$ and φ_2 are proportional by a real positive constant".

In **Section 3** we apply the results of Section 2 to the study of the functor of tempered holomorphic solutions of germs of $\mathcal{D}$-modules on a complex curve. We prove that $\mathscr{S}^t\big(\,\cdot\otimes\mathcal{L}^{1/z^k}\big) : \mathsf{GM}_k \to \mathrm{Mod}(\mathbb{C}_{X_{sa}})$ is a fully faithful functor and that, given a germ of $\mathcal{D}$-module $\mathcal{M}$ with Katz invariant $< k$, the datum of $\mathscr{S}^t\big(\mathcal{M}\otimes\mathcal{L}^{1/z^{k+1}}\big)$ is equivalent to the data of the determinant polynomials of $\mathcal{M}$ and holomorphic solutions of $\mathcal{M}$.

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1 Notations and review

In this section we recall the definitions and the classical results concerning:

(i) subanalytic sets, the subanalytic site and sheaves on it,

(ii) the subanalytic sheaf of tempered holomorphic functions,

(iii) germs of $\mathcal{D}$-modules on a complex curve.

1.1 The subanalytic site

Let M be a real analytic manifold, $\mathcal{A}$ the sheaf of real-valued real analytic functions on M.

Definition 1.1.1 *(i) A set $X \subset M$ is said* semi-analytic at $x \in M$ *if the following condition is satisfied. There exists an open neighborhood W of x such that $X \cap W = \cup_{i\in I} \cap_{j\in J} X_{ij}$ where I and J are finite sets and either $X_{ij} = \{y \in W;\ f_{ij}(y) > 0\}$ or $X_{ij} = \{y \in W;\ f_{ij}(y) = 0\}$ for some $f_{ij} \in \mathcal{A}(W)$. Further, X is said* semi-analytic *if X is semi-analytic at any $x \in M$.*

(ii) A set $X \subset M$ is said subanalytic *if the following condition is satisfied. For any $x \in M$, there exist an open neighborhood W of x, a real analytic manifold N and a relatively compact semi-analytic set $A \subset M \times N$ such that $\pi(A) = X \cap W$, where $\pi : M \times N \to M$ is the projection.*

Given $X \subset M$, denote by $\overset{\circ}{X}$ (resp. $\overline{X}$, ∂X) the interior (resp. the closure, the boundary) of X.

Proposition 1.1.2 (See [BM]) *Let X and Y be subanalytic subset of M. Then $X \cup Y$, $X \cap Y$, $\overline{X}$, $\overset{\circ}{X}$ and $X \setminus Y$ are subanalytic. Moreover the connected components of X are subanalytic, the family of connected components of X is locally finite and X is locally connected.*

Proposition 1.1.3 below is based on Łojasiewicz's inequality, see [BM, Corollary 6.7].

Proposition 1.1.3 *Let $U \subset \mathbb{R}^n$ be an open set, X, Y closed subanalytic subsets of U. For any $x_0 \in X \cap Y$, there exist an open neighborhood W of x_0, $c, r \in \mathbb{R}_{>0}$ such that, for any $x \in W$,*

$$\mathrm{dist}(x, X) + \mathrm{dist}(x, Y) \geq c\,\mathrm{dist}(x, X \cap Y)^r \ .$$

Definition 1.1.4 *Let $\epsilon \in \mathbb{R}_{>0}$, $\gamma :]-\epsilon, \epsilon[\to M$ an analytic map. The set $\gamma(]0, \epsilon[)$ is said a* semi-analytic arc with an endpoint at $\gamma(0)$.

Theorem 1.1.5 (Curve Selection Lemma.) *Let $Z \neq \varnothing$ be a subanalytic subset of M and let $z_0 \in \overline{Z}$. Then there exists an analytic map $\gamma :]-1, 1[\longrightarrow M$, such that $\gamma(0) = z_0$ and $\gamma(t) \in Z$ for $t \neq 0$.*

For the rest of the subsection we refer to [KS01].

Let X be a complex analytic curve, we denote by $\mathrm{Op}(X)$ the family of open subsets of X. For k a commutative unital ring, we denote by $\mathrm{Mod}(k_X)$ the category of sheaves of k-modules on X.

Let us recall the definition of the subanalytic site X_{sa} associated to X. An element $U \in \mathrm{Op}(X)$ is an open set for X_{sa} if it is open, relatively compact and subanalytic in X. The family of open sets of X_{sa} is denoted $\mathrm{Op}^c(X_{sa})$. For $U \in \mathrm{Op}^c(X_{sa})$, a subset S of the family of open subsets of U is said an open covering of U in X_{sa} if $S \subset \mathrm{Op}^c(X_{sa})$ and, for any compact K of X, there exists a finite subset $S_0 \subset S$ such that $K \cap (\cup_{V \in S_0} V) = K \cap U$.

We denote by $\mathrm{Mod}(k_{X_{sa}})$ the category of sheaves of k-modules on the subanalytic site. With the aim of defining the category $\mathrm{Mod}(k_{X_{sa}})$, the adjective "relatively compact" can be omitted in the definition above. Indeed, in [KS01, Remark 6.3.6], it is proved that $\mathrm{Mod}(k_{X_{sa}})$ is equivalent to the category of sheaves on the site whose open sets are the open subanalytic subsets of X and whose coverings are the same as X_{sa}.

Given $Y \in \mathrm{Op}^c(X_{sa})$, we denote by $Y_{X_{sa}}$ the site induced by X_{sa} on Y, defined as follows. The open sets of $Y_{X_{sa}}$ are open subanalytic subsets of Y. A covering of $U \in \mathrm{Op}(Y_{sa})$ for the topology $Y_{X_{sa}}$ is a covering of U is X_{sa}.

We denote by $\varrho : X \longrightarrow X_{sa}$, the natural morphism of sites associated to $\mathrm{Op}^c(X_{sa}) \longrightarrow \mathrm{Op}(X)$. We refer to [KS01] for the definitions of the functors

$\varrho_* : \mathrm{Mod}(k_X) \longrightarrow \mathrm{Mod}(k_{X_{sa}})$ and $\varrho^{-1} : \mathrm{Mod}(k_{X_{sa}}) \longrightarrow \mathrm{Mod}(k_X)$ and for Proposition 1.1.6 below.

Proposition 1.1.6 *(i) ϱ^{-1} is left adjoint to ϱ_*.*

(ii) ϱ^{-1} has a left adjoint denoted by $\varrho_! : \mathrm{Mod}(k_X) \longrightarrow \mathrm{Mod}(k_{X_{sa}})$.

(iii) ϱ^{-1} and $\varrho_!$ are exact and ϱ_* is exact on $\mathbb{R}$-constructible sheaves.

(iv) ϱ_* and $\varrho_!$ are fully faithful.

Through ϱ_*, we will consider $\mathrm{Mod}(k_X)$ as a subcategory of $\mathrm{Mod}(k_{X_{sa}})$.

The functor $\varrho_!$ is described as follows. If $U \in \mathrm{Op}^c(X_{sa})$ and $F \in \mathrm{Mod}(k_X)$, then $\varrho_!(F)$ is the sheaf on X_{sa} associated to the presheaf $U \mapsto F\big(\overline{U}\big)$.

Remark 1.1.7 *It is worth to mention that, given an analytic manifold X, there exists a topological space X' such that the category of sheaves on X_{sa} with values in sets is equivalent to the category of sheaves on X' with values in sets. A detailed description of the semi-algebraic case and the o-minimal case are presented respectively in [BCR] and [Co00].*

1.2 Definition and main properties of $\mathcal{O}^t_{X_{sa}}$

For this subsection we refer to [KS01].

Let X be a complex analytic curve, denote by $\overline{X}$ the complex conjugate curve and by $X_\mathbb{R}$ the underlying real analytic manifold. We denote by X_{sa} the subanalytic site relative to $X_\mathbb{R}$.

Denote by $\mathcal{O}_X$ (resp. $\mathcal{D}_X$) the sheaf of holomorphic functions (resp. linear differential operators with holomorphic coefficients) on X. Denote by $\mathcal{D}b_{X_\mathbb{R}}$ the sheaf of distributions on $X_\mathbb{R}$ and, for a closed subset Z of X, by $\Gamma_Z(\mathcal{D}b_{X_\mathbb{R}})$ the subsheaf of sections supported by Z. One denotes by $\mathcal{D}b^t_{X_{sa}}$ the presheaf of tempered distributions on X_{sa} defined as follows,

$$\mathrm{Op}^c(X_{sa}) \ni U \longmapsto \mathcal{D}b^t_{X_{sa}}(U) := \frac{\Gamma(X; \mathcal{D}b_{X_\mathbb{R}})}{\Gamma_{X\setminus U}(X; \mathcal{D}b_{X_\mathbb{R}})} \ .$$

In [KS01], using some results of [Ło], it is proved that $\mathcal{D}b^t_{X_{sa}}$ is a sheaf on X_{sa}. This sheaf is well defined in the category $\mathrm{Mod}(\varrho_!\mathcal{D}_X)$. Moreover, for any $U \in \mathrm{Op}^c(X_{sa})$, $\mathcal{D}b^t_{X_{sa}}$ is $\Gamma(U, \cdot)$-acyclic.

Denote by $D^b\big(\varrho_!\mathcal{D}_X\big)$ the bounded derived category of $\varrho_!\mathcal{D}_X$-modules. The sheaf $\mathcal{O}^t_{X_{sa}} \in D^b\big(\varrho_!\mathcal{D}_X\big)$ of tempered holomorphic functions is defined as

$$\mathcal{O}^t_{X_{sa}} := R\mathcal{H}om_{\varrho_!\mathcal{D}_{\overline{X}}}\big(\varrho_!\mathcal{O}_{\overline{X}}, \mathcal{D}b^t_{X_\mathbb{R}}\big) \ .$$

In [KS01], it is proved that, since $\dim X = 1$, $R\varrho_*\mathcal{O}_X$ and $\mathcal{O}^t_{X_{sa}}$ are concentrated in degree 0 . Hence we can write the following exact sequence of sheaves on X_{sa}

$$0 \longrightarrow \mathcal{O}^t_{X_{sa}} \longrightarrow \mathcal{D}b^t_{X_{sa}} \xrightarrow{\bar{\partial}} \mathcal{D}b^t_{X_{sa}} \longrightarrow 0 \ .$$

Let us recall that the definition of $\mathcal{D}b^t_{X_{sa}}$ and $\mathcal{O}^t_{X_{sa}}$ can be given without any change in the case of X a complex analytic manifold (see [KS01]).

Now we recall the definition of polynomial growth for $\mathcal{C}^\infty$ functions on $X_\mathbb{R}$ and in (1.2) we give an alternative expression for $\mathcal{O}^t_{X_{sa}}(U)$, $U \in \mathrm{Op}^c(X_{sa})$.

Definition 1.2.1 *Let U be an open subset of X, $f \in \mathcal{C}^\infty_{X_\mathbb{R}}(U)$. One says that f has* polynomial growth at $p \in X$ *if it satisfies the following condition. For a local coordinate system $x = (x_1, x_2)$ around p, there exist a compact neighborhood K of p and $M \in \mathbb{Z}_{>0}$ such that*

$$\sup_{x \in K \cap U} \mathrm{dist}(x, K \setminus U)^M \big| f(x) \big| < +\infty \ . \tag{1.1}$$

We say that $f \in \mathcal{C}^\infty_{X_\mathbb{R}}(U)$ has polynomial growth on U *if it has polynomial growth at any $p \in X$. We say that f is* tempered at p *if all its derivatives have polynomial growth at $p \in X$. We say that f is* tempered on U *if it is tempered at any $p \in X$. Denote by $\mathcal{C}^{\infty,t}_{X_{sa}}$ the presheaf on $X_\mathbb{R}$ of tempered $\mathcal{C}^\infty$-functions.*

It is obvious that f has polynomial growth at any point of U.

In [KS01] it is proved that $\mathcal{C}^{\infty,t}_{X_{sa}}$ is a sheaf on X_{sa}. For $U \subset \mathbb{R}^2$ a relatively compact open set, there is a simple characterization of functions with polynomial growth on U.

Proposition 1.2.2 *Let $U \subset \mathbb{R}^2$ be a relatively compact open set and let $f \in \mathcal{C}^\infty_{\mathbb{R}^2}(U)$. Then f has polynomial growth if and only if there exist $C, M \in \mathbb{R}_{>0}$ such that, for any $x \in U$,*

$$\big| f(x) \big| \leq \frac{C}{\mathrm{dist}(x, \partial U)^M} \ .$$

For Proposition 1.2.3 below, see [KS01].

Proposition 1.2.3 *One has the following isomorphism*

$$\mathcal{O}^t_{X_{sa}} \simeq R\mathcal{H}om_{\varrho_! \mathcal{D}_{\overline{X}}}\big(\varrho_! \mathcal{O}_{\overline{X}}, \mathcal{C}^{\infty,t}_{X_{sa}}\big) \ .$$

Hence, for $U \in \mathrm{Op}^c(X_{sa})$, we deduce the short exact sequence

$$0 \longrightarrow \mathcal{O}^t_{X_{sa}}(U) \longrightarrow \mathcal{C}^{\infty,t}_{X_{sa}}(U) \xrightarrow{\bar{\partial}} \mathcal{C}^{\infty,t}_{X_{sa}}(U) \longrightarrow 0 \ . \tag{1.2}$$

Now, we recall two results on the pull back of tempered holomorphic functions. We refer to [K03] for the definition of $\mathcal{D}_{X \to Y}$, for $f : X \to Y$ a morphism of complex manifolds. For Lemma 1.2.4, see [KS01, Lemma 7.4.7].

Lemma 1.2.4 *Let $f : X \to Y$ be a closed embedding of complex manifolds. There is a natural isomorphism in $D^b(\varrho_!\mathcal{D}_X)$*

$$\varrho_!\mathcal{D}_{X\to Y} \underset{\varrho_! f^{-1}\mathcal{D}_Y}{\overset{L}{\otimes}} f^{-1}\mathcal{O}_Y^t \simeq \mathcal{O}_X^t \ .$$

For Proposition 1.2.5, see [Mo06, Theorem 2.1].

Proposition 1.2.5 *Let $f \in \mathcal{O}_{\mathbb{C}}(X)$, $U \in \mathrm{Op}^c(X_{sa})$ such that $f|_{\overline{U}}$ is an injective map, $h \in \mathcal{O}_X\big(f(U)\big)$. Then $h \in \mathcal{O}^t_{\mathbb{C}_{sa}}\big(f(U)\big)$ if and only if $h \circ f \in \mathcal{O}^t_{X_{sa}}(U)$.*

We conclude this subsection by recalling the definition of the sheaf of holomorphic functions with moderate growth at the origin. We follow [Ma91]. Let S^1 be the unit circle, $S^1 \times \mathbb{R}_{\geq 0}$ the real blow-up at the origin of $\mathbb{C}^\times$.

For $\tau \in \mathbb{R}$, $r \in \mathbb{R}_{>0}$, $0 < \epsilon < \pi$, the set

$$S_{\tau\pm\epsilon,r} := \big\{\varrho e^{i\vartheta} \in \mathbb{C}^\times;\ \varrho \in]0,r[,\ \vartheta \in]\tau-\epsilon,\tau+\epsilon[\big\}$$

is called an *open sector centered at τ of amplitude 2ϵ and radius r* or simply *an open sector*. Identifying S^1 with $[0,2\pi[\subset \mathbb{R}$, we will consider sectors centered at $\tau \in S^1$. Further, with an abuse of language, we will say that an open sector $S_{\tau\pm\epsilon,r}$ contains $\vartheta \in \mathbb{R}$ or $e^{i\vartheta} \in S^1$ if $\vartheta \in]\tau-\epsilon,\tau+\epsilon[\pmod{2\pi}$.

The sheaf on $S^1 \times \mathbb{R}_{\geq 0}$ of holomorphic functions with moderate growth at the origin, denoted $\mathcal{A}^{\leq 0}$, is defined as follows. For U an open set of $S^1 \times \mathbb{R}_{\geq 0}$, set

$$(1.3)\quad \mathcal{A}^{\leq 0}(U) = \left\{ \begin{array}{l} f \in \mathcal{O}_{\mathbb{C}}(U \setminus (S^1 \times \{0\})) \text{ satisfying the following condition:} \\ \text{for any } (e^{i\vartheta},0) \in U \text{ there exist } C, M \in \mathbb{R}_{>0} \text{ and an open} \\ \text{sector } S \subset U \text{ containing } e^{i\vartheta} \text{ such that } \big|f(z)\big| < C|z|^{-M} \\ \text{for any } z \in S \end{array} \right\} .$$

Clearly $\mathcal{A}^{\leq 0}$ is a sheaf on $S^1 \times \mathbb{R}_{\geq 0}$.

In [P], the author defines the functor ν_0^{sa} of specialization at 0 for the sheaves on the subanalytic site. One has that, $\varrho^{-1}\nu_0^{sa}(\mathcal{O}^t_{\mathbb{C}_{sa}}) \simeq \mathcal{A}^{\leq 0}$.

1.3 $\mathcal{D}$-modules on complex curves and Ω-filtered local systems

In this subsection we recall some classical results on germs of $\mathcal{D}_X$-modules on a complex analytic curve X. For a detailed and comprehensive presentation we refer to [Ma91], [K03] and [BV].

Given a complex analytic curve X and $x_0 \in X$, we denote by $\mathcal{O}_X(*x_0)$ (resp. $\mathcal{D}_X(*x_0)$) the sheaf on X of holomorphic functions on $X \setminus \{x_0\}$, meromorphic at x_0 (resp. the sheaf of rings of differential operators of finite order

with coefficients in $\mathcal{O}_X(*x_0)$). We set for short $\mathcal{O}_{*x_0}$ (resp. $\mathcal{D}_{*x_0}$) for the stalk of $\mathcal{O}_X(*x_0)$ (resp. $\mathcal{D}_X(*x_0)$) at x_0. Further, we denote by $\widehat{\mathcal{O}_{*x_0}}$ (resp. $\widehat{\mathcal{D}_{*x_0}}$) the field of formal Laurent power series (resp. the ring of differential operators with coefficients in $\widehat{\mathcal{O}_{*x_0}}$). The ring $\mathcal{O}_{*x_0}$ comes equipped with a natural valuation $v : \mathcal{O}_{*x_0} \to \mathbb{Z} \cup \{+\infty\}$.

By the choice of a local coordinate z near x_0, we can suppose that $X \subset \mathbb{C}$ is an open neighborhood of $x_0 = 0 \in \mathbb{C}$.

The category of holonomic $\mathcal{D}_{*0}$-modules, denoted $\mathrm{Mod}_h(\mathcal{D}_{*0})$, is equivalent to the category of local meromorphic connections.

For $\varphi \in z^{-1}\mathbb{C}[z^{-1}]$, set $\mathcal{L}^\varphi := \mathcal{D}_{*0} \exp(\varphi)$.

For $l \in \mathbb{Z}_{>0}$, let $\mu_l : \mathbb{C} \to \mathbb{C}$, $z \mapsto z^l$. We denote by μ_l^* the inverse image functor for $\mathcal{D}_{*0}$-modules.

Theorems 1.3.1 and 1.3.3 below are cornerstones in the theory of ordinary differential equations. We refer to [Ma91], [Sa02] and [W].

Theorem 1.3.1 (Levelt-Turrittin's Formal Theorem) *Let $\mathcal{M} \in \mathrm{Mod}_h(\mathcal{D}_{*0})$. There exist $l \in \mathbb{Z}_{>0}$, a finite set $\Sigma \subset z^{-1}\mathbb{C}[z^{-1}]$, a family of regular holonomic $\mathcal{D}_{*0}$-modules indexed by Σ, $\{\mathcal{R}_\varphi\}_{\varphi\in\Sigma}$, and an isomorphism in $\mathrm{Mod}(\widehat{\mathcal{D}_{*0}})$*

$$\mu_l^*\mathcal{M} \otimes \widehat{\mathcal{O}_{*0}} \simeq \bigoplus_{\varphi\in\Sigma} \mathcal{L}^\varphi \otimes \mathcal{R}_\varphi \otimes \widehat{\mathcal{O}_{*0}} \ . \tag{1.4}$$

In the litterature (for example [Ma91]) the definition of the Katz invariant of $\mathcal{M} \in \mathrm{Mod}_h(\mathcal{D}_{*0})$ is given from the Newton polygon of $\mathcal{M}$. For sake of simplicity we give an equivalent definition based on the isomorphism (1.4). Clearly, the valuation v induces a map, still denoted v, $z^{-1}\mathbb{C}[z^{-1}] \to \mathbb{Z} \cup \{+\infty\}$.

Definition 1.3.2 *(i) Let $\mathcal{M} \in \mathrm{Mod}_h(\mathcal{D}_{*0})$. Suppose that* (1.4) *is satisfied, l is minimal and $\Sigma \neq \{0\}$. The* Katz invariant *of $\mathcal{M}$ is $\max\limits_{\varphi\in\Sigma}\{\frac{-v(\varphi)}{l}\}$. If $\Sigma = \{0\}$ then the Katz invariant of $\mathcal{M}$ is* 0.

For $k \in \mathbb{Z}_{>0}$, we denote by $\mathrm{Mod}_h(\mathcal{D}_{*0})_k$ the full abelian subcategory of $\mathrm{Mod}_h(\mathcal{D}_{*0})$ whose objects have Katz invariant strictly smaller than k.

(ii) Let $\Sigma \subset z^{-1}\mathbb{C}[z^{-1}]$ be a finite set and $\{\mathcal{R}_\varphi\}_{\varphi\in\Sigma}$ a family of regular holonomic $\mathcal{D}_{*0}$-modules indexed by Σ. A $\mathcal{D}_{*0}$-module isomorphic to $\bigoplus\limits_{\varphi\in\Sigma} \mathcal{L}^\varphi \otimes \mathcal{R}_\varphi$, is said a good model.

We denote by GM_k the full subcategory of $\mathrm{Mod}_h(\mathcal{D}_{*0})_k$ whose objects are good models.

Roughly speaking, Theorem 1.3.3 below says that the formal isomorphism (1.4) is analytic on sufficiently small open sectors.

Let

$$P := \sum_{j=0}^{m} a_j(z)\left(\frac{d}{dz}\right)^j ,$$

where $m \in \mathbb{Z}_{>0}$, $a_j \in \mathcal{O}_{\mathbb{C}}(X)$ and $a_m \neq 0$. Denote by $\mathcal{C}^0_{\mathbb{C}}$ the sheaf of continuous functions on $\mathbb{C}$. For $l \in \mathbb{Z}_{>0}$, S an open sector, $h \in \{1, \ldots, l\}$, let $\zeta_h : S \to \mathbb{C}$ be the l different inverse functions to $z \mapsto z^l$ defined on S.

Theorem 1.3.3 (Hukuhara-Turrittin's Asymptotic Theorem) *There exist a finite set $\Sigma \subset z^{-1}\mathbb{C}[z^{-1}]$, $l, r_\varphi \in \mathbb{Z}_{>0}$ ($\varphi \in \Sigma$) such that for any $\tau \in \mathbb{R}$, there exist an open sector S containing τ, $f_{\varphi kh} \in \mathcal{O}_{\mathbb{C}}(S) \cap \mathcal{C}^0_{\mathbb{C}}\big(\overline{S} \setminus \{0\}\big)$ ($\varphi \in \Sigma$, $k = 1, \ldots, r_\varphi$, $h = 1, \ldots, l$), satisfying*

(i) $\big\{f_{\varphi kh}(z)\exp(\varphi \circ \zeta_h(z));\ \varphi \in \Sigma,\ k = 1, \ldots, r_\varphi,\ h = 1, \ldots, l\big\}$ is a basis of the $\mathbb{C}$-vector space of germs of holomorphic solutions of $Pu = 0$ on S,

(ii) there exist $C, M \in \mathbb{R}_{>0}$ such that, for any $z \in S$,

$$C|z|^M \leq |f_{\varphi kh}(z)| \leq \big(C|z|^M\big)^{-1}$$

($\varphi \in \Sigma,\ k = 1, \ldots, r_\varphi,\ h = 1, \ldots, l$).

As said above, the category $\mathrm{Mod}_h(\mathcal{D}_{*0})$ is equivalent to the category of local meromorphic connections. There already exists a classification of local meromorphic connections by means of the formal structure and the Stokes phenomena. We recall it in the language of Ω-graded and Ω-filtered local systems. We refer to [DMR], [Ma83], [Ma91] and [BV].

Let Ω^1 be the sheaf on $\mathbb{C}^\times$ of 1-forms with holomorphic coefficients. Let Ω be the local system on S^1 defined as follows: for $\vartheta \in S^1$, choose a determination of $z \mapsto z^{1/l}$ near ϑ and set

$$\Omega_\vartheta := \Big\{\sum_{j=1}^{n} a_j z^{-(\frac{j}{l}+1)} dz \in \Omega^1(S);\ a_j \in \mathbb{C},\ l \in \mathbb{Z}_{>0},\ S \text{ an open sector containing } \vartheta\Big\}.$$

The $\mathbb{C}$-vector space Ω_ϑ can be endowed with a partial order. For $\alpha, \beta \in \Omega_\vartheta$, we write $\beta \prec_\vartheta \alpha$, if $\exp\big(\int \beta - \alpha\big) \in \mathcal{A}^{\leq 0}_{(\vartheta,0)}$.

Definition 1.3.4 *An* Ω-graded local system *on an open subset U of S^1 is the data of a finite rank local system $\mathcal{V}$ on U such that for any $\vartheta \in U$, $\mathcal{V}_\vartheta$ is an Ω_ϑ-graded $\mathbb{C}$-vector space, $\mathcal{V}_\vartheta \simeq \bigoplus_{\alpha \in \Omega_\vartheta} \mathcal{V}_{\vartheta,\alpha}$, satisfying the following condition.*

If ϑ and ϑ' are in the same connected component of U, then the grading of $\mathcal{V}_{\vartheta'}$ is induced by the grading of $\mathcal{V}_\vartheta$ by analytic continuation.

A morphism between two Ω-graded local systems $\mathcal{V}_1, \mathcal{V}_2$ is a morphism of local systems $f : \mathcal{V}_1 \to \mathcal{V}_2$ such that, for any $\vartheta \in S^1$, $f_\vartheta : V_{1\vartheta} \to \mathcal{V}_{2\vartheta}$ is a morphism of Ω_ϑ-graded $\mathbb{C}$-vector spaces.

The analytic continuation induces a monodromy action of $\mathbb{Z}$ on Ω_ϑ that we denote by μ_Ω. Given an Ω-graded local system $\mathcal{V}$, the monodromy action of $\mathbb{Z}$ on $\mathcal{V}_\vartheta$ induces an isomorphism $\mathcal{V}_{\vartheta,\alpha} \simeq \mathcal{V}_{\vartheta,\mu_\Omega \alpha}$. Further, for $\vartheta \in S^1$, the functor $\mathcal{V} \mapsto \mathcal{V}_\vartheta$ is an equivalence between the category of Ω-graded local systems and the category of finite dimensional Ω_ϑ-graded $\mathbb{C}$-vector spaces equipped with a $\mathbb{Z}$-action compatible with μ_Ω.

Let us now define Ω-filtered local systems. Roughly speaking, an Ω-filtered local system is obtained by gluing Ω-graded local systems with maps which preserve the Ω-filtration induced by the partial order on Ω_ϑ.

Definition 1.3.5 *An* Ω-filtered local system *is a finite rank local system $\mathcal{V}$ obtained by the following gluing data.*

(i) An open covering $\{U_j\}_{j\in J}$ of S^1 and Ω-graded local systems $\mathcal{V}_j$ on U_j, $\mathcal{V}_{j\vartheta} \simeq \bigoplus_{\alpha\in\Omega_\vartheta} \mathcal{V}_{j\vartheta,\alpha}$.

(ii) A family of isomorphisms of local systems $\varphi_{jk} : \mathcal{V}_k|_{U_j\cap U_k} \xrightarrow{\sim} \mathcal{V}_j|_{U_j\cap U_k}$ such that, for any $\vartheta \in U_j \cap U_k$,

$$\varphi_{jk\vartheta}\big|_{\mathcal{V}_{k\vartheta,\alpha}} : \mathcal{V}_{k\vartheta,\alpha} \longrightarrow \mathcal{V}_{j\vartheta,\alpha} \oplus \Big(\bigoplus_{\beta\prec_\vartheta\alpha} \mathcal{V}_{j\vartheta,\beta} \Big) . \tag{1.5}$$

Remark that composing the morphism (1.5) with the projection on $\mathcal{V}_{j\vartheta,\alpha}$, one obtains an invertible morphism $\mathcal{V}_{k\vartheta,\alpha} \to \mathcal{V}_{j\vartheta,\alpha}$. Thus one defines invertible morphisms of Ω-graded $\mathbb{C}$-vector spaces $\mathcal{V}_{k\vartheta} \to \mathcal{V}_{j\vartheta}$. In this way one defines the functor *gr* from the category of Ω-filtered local systems to the category of Ω-graded local systems.

Thanks to Theorem 1.3.3, for $\mathcal{M} \in \mathrm{Mod}_h(\mathcal{D}_{*0})$, the solutions of $\mathcal{M}$ can be naturally endowed with a structure of Ω-filtered local system. Thus we have a functor, $\mathscr{S}^\Omega$, from the category $\mathrm{Mod}_h(\mathcal{D}_{*0})$ to the category of Ω-filtered local systems.

Definition 1.3.6 *Let $\mathcal{M} \in \mathrm{Mod}_h(\mathcal{D}_{*0})$, the elements of the set*

$$\Omega(\mathcal{M}) := \Big\{ \varphi \in z^{-1/l}\mathbb{C}[z^{-1/l}] \text{ such that } gr_{\frac{d\varphi}{dz}dz}(\mathscr{S}^\Omega(\mathcal{M})_\vartheta) \neq 0 \Big\}$$

are called the determinant polynomials *of $\mathcal{M}$. Further we set*

$$r_{\varphi,\mathcal{M}} := \dim\ gr_{\frac{d\varphi}{dz}dz}\big(\mathscr{S}^\Omega(\mathcal{M})_\vartheta\big) .$$

For the proof of Theorem 1.3.7 below we refer to [Ma91] or [BV].

Theorem 1.3.7 *(i) The functor $\mathscr{S}^\Omega$ from the category $\mathrm{Mod}_h(\mathcal{D}_{*0})$ to the category of Ω-filtered local systems is an equivalence of categories.*

(ii) The functor $\mathscr{S}^\Omega$ restricted to the category $\mathrm{Mod}_h(\widehat{\mathcal{D}_{*0}})$ is an equivalence of categories onto the category of Ω-graded local systems.

2 Tempered growth of exponential functions

This section is subdivided as follows. In the first part we study the family of sets where a function of the form $\exp(\varphi)$, $\varphi \in z^{-1}\mathbb{C}[z^{-1}]$, is tempered. In the second part we use the results of the first part in order to prove that such family determines φ up to a multiplicative positive constant. Throughout this section $X = \mathbb{C}$.

2.1 Sets where exponential functions have tempered growth

For $\varphi \in z^{-1}\mathbb{C}[z^{-1}] \setminus \{0\}$ and $A \in \mathbb{R}_{>0}$, set

$$U_{\varphi,A} := \left\{z \in \mathbb{C}^\times;\ \mathrm{Re}\left(\varphi(z)\right) < A\right\}\ , \tag{2.1}$$

further set $U_{0,A} := \mathbb{C}$.

First we state and prove the analogue of a result of [KS03].

Proposition 2.1.1 *Let $\varphi \in z^{-1}\mathbb{C}[z^{-1}]$ and $U \in \mathrm{Op}^c(X_{sa})$ with $U \neq \varnothing$. The conditions below are equivalent.*

(i) $\exp\left(\varphi\right) \in \mathcal{O}^t_{X_{sa}}(U)$.

(ii) There exists $A \in \mathbb{R}_{>0}$ such that $U \subset U_{\varphi,A}$.

Before proving Proposition 2.1.1, we need the following

Lemma 2.1.2 ([KS03]) *Let $W \neq \varnothing$ be an open subanalytic subset of $\mathbb{P}^1(\mathbb{C})$, $\infty \notin W$. The following conditions are equivalent.*

(i) There exists $A \in \mathbb{R}_{>0}$ such that $\mathrm{Re}\, z < A$, for any $z \in W$.

(ii) The function $\exp(z)$ has polynomial growth on any semi-analytic arc $\Gamma \subset W$ with an endpoint at ∞. That is, for any semi-analytic arc $\Gamma \subset W$ with an endpoint at ∞, there exist $M, C \in \mathbb{R}_{>0}$ such that, for any $z \in \Gamma$,

$$\left|\exp(z)\right| \leq C\left(1 + |z|^2\right)^M\ . \tag{2.2}$$

Proof. Clearly, *(i)*$\Rightarrow$*(ii)*.

Let us prove *(ii)*$\Rightarrow$*(i)*. Set $z := x + iy$ and suppose that x is not bounded on W. There exist $\epsilon, L \in \mathbb{R}_{>0}$ and a real analytic map

$$\begin{array}{rcl} \gamma : [0,\epsilon[& \longrightarrow & \mathbb{P}^1(\mathbb{C}) \\ t & \longmapsto & (x(t), y(t)) \ , \end{array}$$

such that $\gamma(0) = \infty$, $\gamma\big(]0,\epsilon[\big) \subset W$ and $x\big(]0,\epsilon[\big) =]L, +\infty[$. Since γ is analytic, there exist $q \in \mathbb{Q}$, $c \in \mathbb{R}$ and $\mu \in \mathbb{R}_{>0}$ such that, for any $t \in]0,\epsilon[$,

$$\gamma(t) = \Big(x(t)\ ,\ c\,x(t)^q + O\big(x(t)^{q-\mu}\big)\Big) \ .$$

Now, if (2.2) is satisfied, then $\exp(x)$ has polynomial growth in a neighborhood of $+\infty$, which gives a contradiction. □

Proof of Proposition 2.1.1.

Clearly, $(ii) \Rightarrow (i)$.

$(i) \Rightarrow (ii)$. The result is obvious if $\varphi = 0$. Otherwise, we distinguish two cases.

<u>Case 1</u>: suppose $\varphi(z) = \frac{1}{z}$.

Suppose that for any $A \in \mathbb{R}_{>0}$, there exists $z_A \in U$ such that $\mathrm{Re}\left(\frac{1}{z_A}\right) > A$. Then, by Lemma 2.1.2, there exists a semi-analytic arc with an endpoint at 0, $\Gamma \subset U$, such that $\exp\left(\mathrm{Re}\,\frac{1}{z}\right)$ has not polynomial growth on Γ. That is, for any $M, C \in \mathbb{R}_{>0}$, there exists $z_{M,C} \in \Gamma$ satisfying

$$\exp\left(\mathrm{Re}\,\frac{1}{z_{M,C}}\right) \geq \frac{C}{|z_{M,C}|^M} \ .$$

Apply Proposition 1.1.3 with $X = \overline{\Gamma}$ and $Y = \partial U$. There exist an open neighborhood V of 0, $c, r \in \mathbb{R}_{>0}$ such that, for any $z \in \Gamma \cap V$,

$$|z| \leq c\,\mathrm{dist}(z, \partial U)^r \ .$$

Hence,

$$\exp\left(\mathrm{Re}\,\frac{1}{z_{M,C}}\right) \geq \frac{c^{-1}C}{\mathrm{dist}(z_{M,C}, \partial U)^{rM}} \ .$$

It follows that $\exp\left(\frac{1}{z}\right)$ is not tempered on U.

<u>Case 2</u>: suppose $\varphi(z) = \sum\limits_{j=1}^{n} \frac{a_j}{z^j}$, with $n \in \mathbb{Z}_{>0}$ and $a_n \neq 0$.

Let

$$\begin{aligned} \eta(z) &:= \left(\sum_{j=1}^{n} \frac{a_j}{z^j}\right)^{-1} \\ &= \frac{z^n}{\sum\limits_{j=1}^{n} a_j z^{n-j}} \ . \end{aligned}$$

There exists a neighborhood $W \subset \mathbb{C}$ of 0 such that $\eta \in \mathcal{O}_{\mathbb{C}}(W)$. It is well known that a non-constant holomorphic function is locally the composition of a holomorphic isomorphism and a positive integer power of z. Since it is sufficient to prove the result in a neighborhood of 0 and up to finite coverings, we can suppose that $U \subset W$ and $\eta|_{\overline{U}}$ is injective.

Consider the following commutative triangle,

$$\begin{tikzcd} U \arrow[r, "\eta"] \arrow[dr, "\exp(\varphi)"'] & \eta(U) \arrow[d, "\exp(\frac{1}{\zeta})"] \\ & \mathbb{C}\,. \end{tikzcd}$$

Using Proposition 1.2.5 and Case 1, we have that

$$\begin{aligned} \exp(\varphi) \in \mathcal{O}^t(U) &\Leftrightarrow \exp\left(\frac{1}{\zeta}\right) \in \mathcal{O}^t(\eta(U)) \\ &\Leftrightarrow \eta(U) \subset U_{\frac{1}{\zeta},A} \quad \text{for some } A \in \mathbb{R}_{>0} \\ &\Leftrightarrow U \subset U_{\varphi,A} \qquad \text{for some } A \in \mathbb{R}_{>0}\,. \end{aligned}$$

□

Corollary 2.1.3 *Let* $\varphi \in z^{-1}\mathbb{C}[z^{-1}]$. *Let* S *be an open sector of amplitude* 2π, $U \in \mathrm{Op}^c(X_{sa})$, $\varnothing \neq U \subset S$, $\zeta : S \to \mathbb{C}$, *an inverse of* $z \mapsto z^l$. *Then* $\exp\big(\varphi \circ \zeta\big) \in \mathcal{O}^t_{X_{sa}}(U)$ *if and only if there exists* $A \in \mathbb{R}_{>0}$ *such that* $\zeta(U) \subset U_{\varphi,A}$.

In particular, setting

$$U_{\varphi\circ\zeta,A} := \Big\{ z \in S;\ \mathrm{Re}\big(\varphi \circ \zeta(z)\big) < A \Big\}\ ,$$

one has that $\exp(\varphi \circ \zeta) \in \mathcal{O}^t_{X_{sa}}(U)$ *if and only if there exists* $A \in \mathbb{R}_{>0}$ *such that* $U \subset U_{\varphi\circ\zeta,A}$.

Proof. Let $\mu_l(z) := z^l$. Consider the following commutative diagram,

$$\begin{tikzcd} \zeta(U) \arrow[r, "\mu_l"] \arrow[dr, "\exp(\varphi)"'] & U \arrow[d, "\exp(\varphi\circ\zeta)"] \\ & \mathbb{C}\,. \end{tikzcd}$$

By Proposition 1.2.5, we have that $\exp(\varphi \circ \zeta) \in \mathcal{O}^t_{X_{sa}}(U)$ if and only if $\exp(\varphi) \in \mathcal{O}^t_{X_{sa}}(\zeta(U))$. Then the conclusion follows by Proposition 2.1.1.

□

Now we are going to introduce a class of subanalytic sets which plays an important role in what follows.

Definition 2.1.4 *For $\tau \in \mathbb{R}$ we say that $U \in \mathrm{Op}^c(X_{sa})$ is* concentrated along τ *if $U \neq \varnothing$ is connected, $0 \in \partial U$ and, for any open sector S containing τ, there exists an open neighborhood $W \subset \mathbb{C}$ of 0 such that $U \cap W \subset S$.*

Lemma 2.1.5 below follows easily from the well known fact that a non-constant holomorphic function is locally the composition of a holomorphic isomorphism and a positive integer power of z.

Lemma 2.1.5 *Let $W \subset \mathbb{C}$ be an open neighborhood of 0, $f \in \mathcal{O}(W)$. Suppose that f has a zero of order $l \in \mathbb{Z}_{>0}$ at 0. There exists $\tau_f \in \mathbb{R}$, depending only on the argument of $f^{(l)}(0)$, satisfying the following conditions.*

(i) For any $\tau \in \mathbb{R}$, $U \in \mathrm{Op}^c(X_{sa})$ concentrated along τ, there exists an open neighborhood W' of 0, $\overline{W'} \subset W$, such that $f|_{\overline{U \cap W'}}$ is injective and $f(U \cap W')$ is concentrated along $l(\tau + \tau_f)$.

(ii) For any $\tau \in \mathbb{R}$, $V \in \mathrm{Op}^c(X_{sa})$ concentrated along τ, there exist an open neighborhood W' of 0, $V_0, \ldots, V_{l-1} \in \mathrm{Op}^c(X_{sa})$, such that such that V_j is concentrated along $\frac{\tau}{l} - \tau_f + j\frac{2\pi}{l}$, and $f(V_j) = V \cap W'$ $(j = 0, \ldots, l-1)$.

Proposition 2.1.6 below will play a fundamental role in the next subsection.

Proposition 2.1.6 *Let $n \in \mathbb{Z}_{\geq 0}$, $\tau_0 \in \mathbb{R}$. There exists $\tau \in \mathbb{R}$ such that, for any $\varphi = \frac{\varrho e^{i\tau_0}}{z^n} + \widetilde{\varphi} \in z^{-1}\mathbb{C}[z^{-1}]$ $(\varrho \in \mathbb{R}_{>0}, \ -v(\widetilde{\varphi}) < n)$, there exist $U_0, \ldots, U_{2n-1} \in \mathrm{Op}^c(X_{sa})$ satisfying*

(i) U_j is concentrated along $\tau + j\frac{\pi}{n}$,

(ii) $\exp\big(\varphi\big), \exp\big(-\varphi\big) \in \mathcal{O}^t_{X_{sa}}(U_j)$,

$(j = 0, \ldots, 2n-1)$.

Proof. The result is obvious if $\varphi = 0$. Otherwise we distinguish three cases.
<u>Case 1</u>: suppose $\varphi(z) = \frac{1}{z}$.
Recall (2.1). For $A \in \mathbb{R}_{>0}$, one checks easily that the set $U_{\frac{1}{z},A}$ (resp. $U_{-\frac{1}{z},A}$) is the complementary of the closed ball of center $\big(\frac{1}{2A}, 0\big)$ (resp. $\big(-\frac{1}{2A}, 0\big)$) and radius $\frac{1}{2A}$.
Set

$$\begin{aligned} U_1 &:= \left\{(x,y) \in \mathbb{R}^2;\ |x| < 1,\ \sqrt{|x| - x^2} < y < 1\right\}, \\ U_2 &:= \left\{(x,y) \in \mathbb{R}^2;\ |x| < 1,\ -1 < y < -\sqrt{|x| - x^2}\right\}. \end{aligned}$$

It is easy to see that U_1 (resp. U_2) is concentrated along $\frac{\pi}{2}$ (resp. $\frac{3\pi}{2}$) and $U_1 \cup U_2 \subset U_{\frac{1}{z},1} \cap U_{-\frac{1}{z},1}$. Hence, by Proposition 2.1.1,

$$\exp(1/z),\ \exp(-1/z) \in \mathcal{O}^t_{X_{sa}}(U_j) \qquad (j=1,2)\ . \tag{2.3}$$

<u>Case 2</u>: suppose that $\varphi(z) = \frac{1}{z^m}$, for $m \in \mathbb{Z}_{>0}$. Let $\mu_m : \mathbb{C} \to \mathbb{C}$, $\mu_m(z) = z^m$. Consider the commutative triangle

$$\begin{tikzcd} \mathbb{C}^\times \arrow[r, "\mu_m"] \arrow[dr, "\exp(1/z^m)"'] & \mathbb{C}^\times \arrow[d, "\exp(1/z)"] \\ & \mathbb{C}\ . \end{tikzcd}$$

Consider U_1, U_2 as in Case 1. Applying Lemma 2.1.5 *(ii)* with $f = \mu_m$, we obtain that there exist $V_{1,0}, \dots, V_{1,m-1} \in \mathrm{Op}^c(X_{sa})$ (resp. $V_{2,0}, \dots, V_{2,m-1} \in \mathrm{Op}^c(X_{sa})$) such that

(i) $V_{1,j}$ (resp. $V_{2,j}$) is concentrated along $\frac{\pi}{2m} + j\frac{2\pi}{m}$ (resp. $\frac{3\pi}{2m} + j\frac{2\pi}{m}$),

(ii) $\mu_m(V_{k,j}) = U_k$,

$(j = 0, \dots, m-1,\ k = 1,2)$.

Clearly, $\mu_m|_{\overline{V_{k,j}}}$ is injective. By Proposition 1.2.5, we have that

$$\exp\big(1/z^m\big) \in \mathcal{O}^t_{X_{sa}}(V_{k,j})$$

$$\Big(\text{resp. } \exp\big(-1/z^m\big) \in \mathcal{O}^t_{X_{sa}}(V_{k,j})\Big)$$

if and only if

$$\exp\big(1/z\big) \in \mathcal{O}^t_{X_{sa}}\big(\mu_m(V_{k,j})\big) = \mathcal{O}^t_{X_{sa}}(U_k)$$

$$\Big(\text{resp. } \exp\big(-1/z\big) \in \mathcal{O}^t_{X_{sa}}\big(\mu_m(V_{k,j})\big) = \mathcal{O}^t_{X_{sa}}(U_k)\Big)$$

$(j = 0, \dots, m-1, k = 1,2)$. The conclusion follows.

<u>Case 3</u>: suppose that

$$\varphi(z) = \sum_{j=1}^{n} \frac{a_j}{z^j} \in z^{-1}\mathbb{C}[z^{-1}]\ ,$$

for $n \in \mathbb{Z}_{>0}$, $a_j \in \mathbb{C}$ $(j = 1, \dots, n)$ and $a_n \neq 0$.

First, we recall the implicit function theorem for convergent power series. We denote by $\mathbb{C}\{x\}$ (resp. $\mathbb{C}\{x,y\}$) the ring of convergent power series in x (resp. x, y). We refer to [C, Theorem 8.6.1, p.166] for the proof.

Theorem 2.1.7 *Let $F \in \mathbb{C}\{x,y\}$ be such that $F(0,0) = 0$. There exists $\eta(x) \in \cup_{l \in \mathbb{Z}_{>0}} x^{1/l}\mathbb{C}\{x^{1/l}\}$ such that $F(x, \eta(x)) = 0$.*

Consider

$$
\begin{aligned}
F(z,\eta) &:= -\eta^n + z^n \sum_{j=1}^{n} a_j \eta^{n-j} \\
&= -\eta^n + z^n a_1 \eta^{n-1} + \ldots + z^n a_{n-1}\eta + z^n a_n \ .
\end{aligned} \tag{2.4}
$$

By Theorem 2.1.7, there exist $l \in \mathbb{Z}_{>0}$, $\eta(z) \in z^{1/l}\mathbb{C}\{z^{1/l}\}$ such that $F\big(z,\eta(z)\big) = 0$. Since $a_n \neq 0$, we have that $\eta(z) \neq 0$, for $z \neq 0$. It follows that $\eta(z) \in z^{1/l}\mathbb{C}\{z^{1/l}\}$ satisfies

$$
\varphi\big(\eta(z)\big) = \sum_{j=1}^{n} \frac{a_j}{\eta(z)^j} = \frac{1}{z^n} \ . \tag{2.5}
$$

Further, substituting $\eta(z)$ in (2.4), one checks that $l = 1$ and $\eta(z) = z\sigma(z)$, for σ an invertible element of $\mathbb{C}\{z\}$ such that $\arg\big(\sigma(0)\big) = \frac{\arg(a_n)}{n}$. In particular, there exists an open neighborhood $W \subset \mathbb{C}$ of the origin such that $\eta \in \mathcal{O}_{\mathbb{C}}(W)$.

Now, by Case 2, there exist $V_{k,j} \subset W$ $(j = 0, \ldots, n-1,\ k = 1,2)$ such that $V_{1,j}$ (resp. $V_{2,j}$) is concentrated along $\frac{\pi}{2n} + j\frac{2\pi}{n}$ (resp. $\frac{3\pi}{2n} + j\frac{2\pi}{n}$) and

$$
\exp\Big(\frac{1}{z^n}\Big), \exp\Big(-\frac{1}{z^n}\Big) \in \mathcal{O}^t_{X_{sa}}(V_{k,j}) \tag{2.6}
$$

$(j = 0, \ldots, n-1,\ k = 1,2)$.

As η has a zero of order 1 at 0, by Lemma 2.1.5 *(i)*, there exists $\tau_\eta \in \mathbb{R}$, depending only on $\arg\big(\eta(0)\big) = \frac{\arg(a_n)}{n}$, such that, up to shrinking W,

(i) $\eta|_{\overline{V_{k,j}}}$ is injective and

(ii) $\eta\big(V_{1,j}\big)$ (resp. $\eta\big(V_{2,j}\big)$) is concentrated along $\tau_\eta + \frac{\pi}{2n} + j\frac{2\pi}{n}$ (resp. $\tau_\eta + \frac{3\pi}{2n} + j\frac{2\pi}{n}$),

$(j = 0, \ldots, n-1)$.

Consider the commutative triangle

$$
\begin{tikzcd}
W \setminus \{0\} \arrow[r, "\eta"] \arrow[dr, "\exp\left(\frac{1}{z^n}\right)"'] & \mathbb{C}^\times \arrow[d, "\exp(\varphi(z))"] \\
& \mathbb{C} \ .
\end{tikzcd}
$$

By Proposition 1.2.5, we have that

$$
\exp\big(\varphi(z)\big) \in \mathcal{O}^t_{X_{sa}}\big(\eta(V_{k,j})\big)
$$

$$
\Big(\text{resp. } \exp\big(-\varphi(z)\big) \in \mathcal{O}^t_{X_{sa}}\big(\eta(V_{k,j})\big)\Big)
$$

if and only if

$$\exp\big(\varphi\circ\eta(z)\big) = \exp\big(1/z^n\big) \in \mathcal{O}^t_{X_{sa}}\big(V_{k,j}\big)$$

$$\Big(\text{resp. } \exp\big(-\varphi\circ\eta(z)\big) = \exp\big(-1/z^n\big) \in \mathcal{O}^t_{X_{sa}}\big(V_{k,j}\big)\Big)$$

$(j = 0,\dots,n-1, k = 1,2)$. The conclusion follows from (2.6).

□

Remark 2.1.8 Recall the definition given in the end of Subsection 1.2 of the sheaf $\mathcal{A}^{\leq 0}$ defined on $S^1\times\mathbb{R}_{\geq 0}$, considered as the real blow-up at 0 of $\mathbb{C}^\times$. Let $\tau\in\mathbb{R}$, $U\in\mathrm{Op}^c(X_{sa})$ concentrated along τ, the set $(\tau,0)\cup U\subset S^1\times\mathbb{R}_{\geq 0}$ is not open. Further if $\exp(\varphi)\in\mathcal{A}^{\leq 0}_{(\tau,0)}$ then $\exp(-\varphi)\notin\mathcal{A}^{\leq 0}_{(\tau,0)}$.

We conclude this subsection with an easy lemma which will be useful in the next subsection. First, let us introduce some notation.

Given $\varphi\in z^{-1}\mathbb{C}[z^{-1}]$, $\varphi = \frac{\eta e^{i\tau}}{z^n}+\tilde\varphi$, for $\eta\in\mathbb{R}_{>0}, n\in\mathbb{Z}_{>0}, \tau\in\mathbb{R}$ and $\tilde\varphi\in z^{-1}\mathbb{C}[z^{-1}]$, $-v(\tilde\varphi)<n$, set

$$I_\varphi := \Big\{\vartheta\in[0,2\pi];\ \cos(\tau-n\vartheta)<0\Big\}\ .$$

In other words, I_φ is the support of $\exp(\varphi)$ as a section of $\mathcal{A}^{\leq 0}\big|_{S^1\times\{0\}}$.

Recall the definition of the sets $U_{\varphi,A}$ given in (2.1).

Lemma 2.1.9 *(i) Let $\varphi_1,\varphi_2\in z^{-1}\mathbb{C}[z^{-1}]$, $\varphi_j := \frac{\eta_j e^{i\tau_j}}{z^{n_j}}+\tilde\varphi_j$, for $\eta_j\in\mathbb{R}_{>0}, n_j\in\mathbb{Z}_{>0}, \tau_j\in\mathbb{R}$ and $\tilde\varphi_j\in z^{-1}\mathbb{C}[z^{-1}]$, $-v(\tilde\varphi_j)<n_j$.*

If $n_1\neq n_2$ or $\tau_1\neq\tau_2$, then $I_{\varphi_1}\setminus\overline{I}_{\varphi_2}\neq\varnothing$ and $I_{\varphi_2}\setminus\overline{I}_{\varphi_1}\neq\varnothing$.

(ii) Let $\vartheta_0\in[0,2\pi[$ and $\varphi\in z^{-1}\mathbb{C}[z^{-1}]\setminus\{0\}$. If $\vartheta_0\in I_\varphi$, then there exists an open sector S containing ϑ such that, for any $A\in\mathbb{R}_{>0}$, $S\subset U_{\varphi,A}$. In particular, for any $U\in\mathrm{Op}^c(X_{sa})$ concentrated along ϑ_0, $\exp(\varphi)\in\mathcal{O}^t_{X_{sa}}(U)$.

(iii) Let $\vartheta_0\in[0,2\pi[$ and $\varphi\in z^{-1}\mathbb{C}[z^{-1}]\setminus\{0\}$. If $\vartheta_0\notin\overline{I}_\varphi$, then there exists an open sector S containing ϑ such that, for any $A\in\mathbb{R}_{>0}$, $S\subset X\setminus U_{\varphi,A}$. In particular, for any $U\in\mathrm{Op}^c(X_{sa})$ concentrated along ϑ_0, $\exp(\varphi)\notin\mathcal{O}^t_{X_{sa}}(U)$.

Proof.

The result follows from some easy computations.

□

2.2 Comparison between growth of exponential functions

In this subsection we are going to use the results of the previous subsection in order to prove that if $\varphi_1, \varphi_2 \in z^{-1}\mathbb{C}[z^{-1}]$ and, for any $\lambda \in \mathbb{R}_{>0}$, $\varphi_1 \neq \lambda\varphi_2$, then the families $\{U_{\varphi_1,A}\}_{A\in\mathbb{R}_{>0}}$ and $\{U_{\varphi_2,A}\}_{A\in\mathbb{R}_{>0}}$ are not cofinal.

The main result of this subsection is Proposition 2.2.1 below.

Proposition 2.2.1 *Let $\varphi_1, \varphi_2 \in z^{-1}\mathbb{C}[z^{-1}] \setminus \{0\}$.*

(i) Suppose that there exists $\lambda \in \mathbb{R}_{>0}$ such that $\varphi_1 = \lambda\varphi_2$. Then, for any $A \in \mathbb{R}_{>0}$, $U_{\varphi_1,A} = U_{\varphi_2,\frac{A}{\lambda}}$. In particular, for any $U \in \mathrm{Op}^c(X_{sa})$, $\exp(\varphi_1) \in \mathcal{O}^t_{X_{sa}}(U)$ if and only if $\exp(\varphi_2) \in \mathcal{O}^t_{X_{sa}}(U)$.

(ii) Suppose that, for any $\lambda \in \mathbb{R}_{>0}$, $\varphi_1 \neq \lambda\varphi_2$. Then for any open sector S of amplitude $> \frac{2\pi}{\max\{-v(\varphi_1),-v(\varphi_2),2\}}$ at least one of the two conditions below is satisfied (resp. for any open neighborhood S of 0, both conditions below are satisfied).

(a) There exists $U \in \mathrm{Op}^c(X_{sa})$, $U \subset S$ such that $\exp(\varphi_1) \in \mathcal{O}^t_{X_{sa}}(U)$ and $\exp(\varphi_2) \notin \mathcal{O}^t_{X_{sa}}(U)$.

(b) There exists $V \in \mathrm{Op}^c(X_{sa})$, $V \subset S$ such that $\exp(\varphi_1) \notin \mathcal{O}^t_{X_{sa}}(V)$ and $\exp(\varphi_2) \in \mathcal{O}^t_{X_{sa}}(V)$.

Proof.
(i) Obvious.
(ii) For S an open sector, set $\tilde{S} := \left\{\vartheta \in [0, 2\pi[; \exists\, r > 0\ re^{i\vartheta} \in S\right\}$.
Let

$$\varphi_1(z) := \frac{\eta_1 e^{i\tau_1}}{z^{n_1}} + \tilde{\varphi}_1(z) \qquad \text{and} \qquad \varphi_2(z) := \frac{\eta_2 e^{i\tau_2}}{z^{n_2}} + \tilde{\varphi}_2(z)\ ,$$

for $\eta_j \in \mathbb{R}_{>0}$, $\tau_j \in [0, 2\pi[$ and $\tilde{\varphi}_j(z) \in z^{-1}\mathbb{C}[z^{-1}]$, $-v(\tilde{\varphi}_j) < n_j$ $(j = 1, 2)$.

Suppose that $n_1 \neq n_2$ or $\tau_1 \neq \tau_2$.

By Lemma 2.1.9 *(i)*, $I_{\varphi_1} \setminus \overline{I}_{\varphi_2} \neq \varnothing$ and $I_{\varphi_2} \setminus \overline{I}_{\varphi_1} \neq \varnothing$.

Let S be an open neighborhood of 0, $\vartheta_1 \in I_{\varphi_1} \setminus \overline{I}_{\varphi_2}$ and $\vartheta_2 \in I_{\varphi_2} \setminus \overline{I}_{\varphi_1}$. There exist $U_k \in \mathrm{Op}^c(X_{sa})$ concentrated along ϑ_k such that $U_k \subset S$ $(k = 1, 2)$. The result follows by Lemma 2.1.9 *(ii),(iii)*.

Suppose that S is an open sector of amplitude $> \frac{2\pi}{\max\{n_1,n_2,2\}}$. Then, there exists $\vartheta \in \tilde{S}$ such that either $\vartheta \in I_{\varphi_1} \setminus \overline{I}_{\varphi_2}$ or $\vartheta \in I_{\varphi_2} \setminus \overline{I}_{\varphi_1}$. Since $\vartheta \in \tilde{S}$, there exists $U \in \mathrm{Op}^c(X_{sa})$ concentrated along ϑ such that $U \subset S$. The conclusion follows by Lemma 2.1.9 *(ii),(iii)*.

Now suppose that $n_1 = n_2 = n$ and $\tau_1 = \tau_2$. That is,

$$\varphi_1(z) = \frac{\eta_1 e^{i\tau_1}}{z^{n}} + \tilde{\varphi}_1(z) \qquad \text{and} \qquad \varphi_2(z) = \frac{\eta_2 e^{i\tau_1}}{z^{n}} + \tilde{\varphi}_2(z)\ .$$

Since, for any $\lambda \in \mathbb{R}_{>0}$, $\varphi_1 \neq \lambda\varphi_2$, we have that $n \geq 2$.

Set $\psi_{21} := \varphi_2 - \frac{\eta_2}{\eta_1}\varphi_1$ and $\psi_{12} := \varphi_1 - \frac{\eta_1}{\eta_2}\varphi_2$. Since $\psi_{21} \neq 0$ and $\psi_{21} = -\frac{\eta_2}{\eta_1}(\varphi_1 - \frac{\eta_1}{\eta_2}\varphi_2) = -\frac{\eta_2}{\eta_1}\psi_{12}$, then $I_{\psi_{21}} = I_{-\psi_{12}}$.

By Proposition 2.1.6, there exist $\tau \in \mathbb{R}$ and $U_0, \dots, U_{2n-1}, V_0, \dots, V_{2n-1} \in \mathrm{Op}^c(X_{sa})$ satisfying the conditions

(i) U_j, V_j are concentrated along $\tau + j\frac{\pi}{n}$,

(ii) $\exp(\varphi_1), \exp(-\varphi_1) \in \mathcal{O}^t_{X_{sa}}(U_j)$ and $\exp(\varphi_2), \exp(-\varphi_2) \in \mathcal{O}^t_{X_{sa}}(V_j)$,

$(j = 0, \dots, 2n-1)$.

Since $-v(\psi_{12}) = -v(\psi_{21}) < n$, if S is an open sector of amplitude $> \frac{2\pi}{n}$, there exists $j' \in \{0, \dots, 2n-1\}$ such that $\tau + j'\frac{\pi}{n} \in \tilde{S}$ and either $\tau + j'\frac{\pi}{n} \notin \overline{I}_{\psi_{12}}$ or $\tau + j'\frac{\pi}{n} \notin \overline{I}_{\psi_{21}}$.

More generically, $\{\tau + j\frac{\pi}{n};\ j \in 0, \dots, 2n-1\} \not\subset \overline{I}_{\psi_{12}}$ and $\{\tau + j\frac{\pi}{n};\ j \in 0, \dots, 2n-1\} \not\subset \overline{I}_{\psi_{21}}$.

Let us consider the case $\tau + j'\frac{\pi}{n} \notin \overline{I}_{\psi_{12}}$. Since $V_{j'}$ is concentrated along $\tau + j'\frac{\pi}{n}$, Lemma 2.1.9 *(iii)* implies

$$\exp(\psi_{12}) \notin \mathcal{O}^t_{X_{sa}}(V_{j'}) \ . \tag{2.7}$$

Suppose now that $\exp(\varphi_1) \in \mathcal{O}^t_{X_{sa}}(V_{j'})$. Since $\exp(-\varphi_2), \exp(\varphi_2) \in \mathcal{O}^t_{X_{sa}}(V_{j'})$ and the product of tempered functions is tempered, we have that $\exp(\varphi_1 - \frac{\eta_1}{\eta_2}\varphi_2) \in \mathcal{O}^t_{X_{sa}}(V_{j'})$, which contradicts (2.7). Hence $\exp(\varphi_1) \notin \mathcal{O}^t_{X_{sa}}(V_{j'})$.

Let us consider the case $\tau + j'\frac{\pi}{n} \notin \overline{I}_{\psi_{21}}$. Since $U_{j'}$ is concentrated along $\tau + j'\frac{\pi}{n}$, Lemma 2.1.9 *(iii)* implies

$$\exp(\psi_{21}) \notin \mathcal{O}^t_{X_{sa}}(U_{j'}) \ . \tag{2.8}$$

Suppose now that $\exp(\varphi_2) \in \mathcal{O}^t_{X_{sa}}(U_{j'})$. Since $\exp(\varphi_1), \exp(-\varphi_1) \in \mathcal{O}^t_{X_{sa}}(U_{j'})$ and the product of tempered functions is tempered, we have that $\exp(\varphi_2 - \frac{\eta_2}{\eta_1}\varphi_1) \in \mathcal{O}^t_{X_{sa}}(U_{j'})$, which contradicts (2.8). Hence $\exp(\varphi_2) \notin \mathcal{O}^t_{X_{sa}}(U_{j'})$. □

Corollary 2.2.2 *Let $l \in \mathbb{Z}_{>0}$ $\omega, \varphi_1, \varphi_2 \in z^{-1}\mathbb{C}[z^{-1}]$, such that $-v(\omega) > \max_{j=1,2}\{\frac{-v(\varphi_j)}{l} + 1\}$. Let S an open sector of amplitude 2π, ζ an inverse of $z \mapsto z^l$ defined on S. The following conditions are equivalent.*

(i) $\varphi_1 \circ \zeta \neq \varphi_2 \circ \zeta$.

(ii) At least one of the following two conditions is verified:

(a) there exists $U \in \mathrm{Op}^c(X_{sa})$, $U \subset S$ such that $\exp(\varphi_1 \circ \zeta + \omega) \in \mathcal{O}^t_{X_{sa}}(U)$ and $\exp(\varphi_2 \circ \zeta + \omega) \notin \mathcal{O}^t_{X_{sa}}(U)$;

(b) *there exists* $V \in \mathrm{Op}^c(X_{sa})$*,* $V \subset S$ *such that* $\exp(\varphi_1 \circ \zeta + \omega) \notin \mathcal{O}^t_{X_{sa}}(V)$ *and* $\exp(\varphi_2 \circ \zeta + \omega) \in \mathcal{O}^t_{X_{sa}}(V)$*.*

Proof. (ii)⇒(i). Obvious.

(i)⇒(ii). Set $\mu_l(z) := z^l$.

Suppose now that $\varphi_1 \circ \zeta \neq \varphi_2 \circ \zeta$. It follows that, for any $\lambda \in \mathbb{R}_{>0}$, $\lambda(\varphi_1 + \omega \circ \mu_l) \neq \varphi_2 + \omega \circ \mu_l$. Consider the sector $\zeta(S)$ of amplitude $\frac{2\pi}{l}$. Since $-v(\omega) \geq 2$, then $\frac{2\pi}{l} > -\frac{2\pi}{lv(\omega)}$. Hence by Proposition 2.2.1 there exists $\zeta(U) \subset \zeta(S)$ such that either $\exp(\varphi_1 + \omega \circ \mu_l) \in \mathcal{O}^t_{X_{sa}}(\zeta(U))$ and $\exp(\varphi_2 + \omega \circ \mu_l) \notin \mathcal{O}^t_{X_{sa}}(\zeta(U))$ or viceversa. By Proposition 1.2.5, it follows that either $\exp(\varphi_1 \circ \zeta + \omega) \in \mathcal{O}^t_{X_{sa}}(U)$ and $\exp(\varphi_2 \circ \zeta + \omega) \notin \mathcal{O}^t_{X_{sa}}(U)$ or viceversa.

□

Remark 2.2.3 There is another way to prove Proposition 2.2.1. Let us briefly summarize it.

Let $\varphi_1, \varphi_2 \in z^{-1}\mathbb{C}[z^{-1}] \setminus \{0\}$, such that, for any $\lambda \in \mathbb{R}_{>0}$, $\varphi_1 \neq \lambda\varphi_2$. Let $n_k = -v(\varphi_k)$ $(k = 1, 2)$.

Let $C_{\varphi_k,A}$ be the boundary of the set $U_{\varphi_k,A}$ $(k = 1, 2)$. One has that $C_{\varphi_k,A}$ is the set of the zeros of a polynomial $Q_{\varphi_k,A}(x, y) \in \mathbb{R}[x, y]$ $(k = 1, 2)$. Further $C_{\varphi_k,A}$ has $2n_k$ distinct branches at 0 determined by the Puiseux's series $\sigma_{\varphi_k,A,1}(x), \ldots, \sigma_{\varphi_k,A,2n_k}(x)$ obtained by solving the equation $Q_{\varphi_k,A}(x, y) = 0$ $(k = 1, 2)$ with respect to y.

One checks that the first n_k terms of $\sigma_{\varphi_k,A,j}(x)$ do not depend on A $(k = 1, 2,\ j = 1, \ldots, 2n_k)$. Further, it turns out that there exists $\vartheta_k \in [0, 2\pi[$ such that the tangent at 0 of the graph of $\sigma_{\varphi_k,A,j}(x)$ has slope $\tan\left(\vartheta_k + j\frac{\pi}{2n_k}\right)$ $(k = 1, 2,\ j = 1, \ldots, 2n_k)$.

If $n_1 \neq n_2$ or $\vartheta_1 \neq \vartheta_2$, the result follows easily.

If $n_1 = n_2$ and $\vartheta_1 = \vartheta_2$, one checks that there exist $\overline{j} \in \{1, \ldots, 2n_1\}$ and $r \in \{1, \ldots, n_1\}$ such that the r-th coefficients of $\sigma_{\varphi_1,A,\overline{j}}(x)$ and $\sigma_{\varphi_2,A,\overline{j}}(x)$ are different. Hence, there are infinitely many relatively compact subanalytic open sets concentrated along some $\vartheta_1 + j\frac{\pi}{2n_1}$ fitting between $\sigma_{\varphi_1,A,\overline{j}}(x)$ and $\sigma_{\varphi_2,A,\overline{j}}(x)$. Choosing U among these sets, one obtains that one exponential is tempered on U and the other is not.

This procedure is more intuitive than the proof we chose to expose here but it is more technical and much longer.

3 Tempered solutions and Ω-filtered local systems

In the first part of this section we are going to prove that the tempered solutions induce a fully faithful functor on good models. In the second part we will prove that the datum of tempered solutions of a meromorphic connection $\mathcal{M}$ is equivalent to the data of determinant polynomials and holomorphic solutions of $\mathcal{M}$.

Let us recall that M. Kashiwara, in [K84], proves that, given a complex analytic manifold X and an object $\mathcal{M}$ of the bounded derived category of $\mathcal{D}_X$-modules with regular holonomic cohomology,

$$R\mathcal{H}\mathrm{om}_{\varrho_!\mathcal{D}_X}(\varrho_!\mathcal{M}, \mathcal{O}^t_{X_{sa}}) \simeq R\mathcal{H}\mathrm{om}_{\varrho_!\mathcal{D}_X}(\varrho_!\mathcal{M}, \mathcal{O}_X) \ .$$

Given a complex analytic curve Y, $x_0 \in Y$, $\mathcal{M} \in \mathrm{Mod}_h(\mathcal{D}_{*x_0})$, there exists a neighborhood $X \subset Y$ of x_0 such that $\mathcal{M}$ is a holonomic $\mathcal{D}_X$-module. By chosing a local coordinate z near x_0, we can suppose that $X = \mathbb{C}$ and $x_0 = 0$. Recall that for $\omega \in z^{-1}\mathbb{C}[z^{-1}]$, we set $\mathcal{L}^\omega := \mathcal{D}_\mathbb{C} \exp(\omega)$.

Set

$$\begin{aligned} \mathscr{S}\big(\mathcal{M}\big) &:= \mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) \ , \\ \mathscr{S}^t\big(\mathcal{M}\big) &:= \mathcal{H}\mathrm{om}_{\varrho_!\mathcal{D}_X}(\varrho_!\mathcal{M}, \mathcal{O}^t_{X_{sa}}) \ , \\ \mathscr{S}^t_\omega(\mathcal{M}) &:= \mathcal{H}\mathrm{om}_{\varrho_!\mathcal{D}_X}(\varrho_!(\mathcal{M} \otimes \mathcal{L}^\omega), \mathcal{O}^t_{X_{sa}}) \ , \end{aligned}$$

the functors ϱ_* and $\varrho_!$ being defined in Subsection 1.1.

3.1 Tempered solutions and non-ramified Ω-graduations

The main results of this subsection are Proposition 3.1.2 and Theorem 3.1.5 below. First, let us describe explicitely the subanalytic sheaf $\mathscr{S}^t\big(\mathcal{L}^\varphi \otimes \mathcal{R}\big)$, for $\varphi \in z^{-1}\mathbb{C}[z^{-1}]$ and $\mathcal{R}$ a regular holonomic $\mathcal{D}_{*0}$-module.

Recall the definition of the sets $U_{\varphi,A}$ given in (2.1).

Lemma 3.1.1 *Let $\varphi \in z^{-1}\mathbb{C}[z^{-1}]$, $\mathcal{R}$ a regular holonomic $\mathcal{D}_{*0}$-module. Then*

$$\mathscr{S}^t\big(\mathcal{L}^\varphi \otimes \mathcal{R}\big) \simeq \varinjlim_{A>0} \varrho_* \ \mathscr{S}\big(\mathcal{R}\big)_{U_{\varphi,A}}$$

Proof. If $\varphi = 0$, the result follows from the fact that $\mathscr{S}^t(\mathcal{R}) \simeq \mathscr{S}(\mathcal{R})$.

Suppose $\varphi \neq 0$. Let $V \in \mathrm{Op}^c(X_{sa})$ be connected and simply connected. If $0 \in V$, then clearly $\Gamma\big(V, \mathscr{S}^t\big(\mathcal{L}^\varphi \otimes \mathcal{R}\big)\big) \simeq 0$. Otherwise, the $\mathbb{C}$-vector space $\Gamma\big(V, \mathscr{S}\big(\mathcal{L}^\varphi \otimes \mathcal{R}\big)\big)$ has finite dimension r and is generated by $h_1(z)\exp(\varphi(z))$, $\ldots$, $h_r(z)\exp(\varphi(z))$, for $h_1, \ldots, h_r \in \mathcal{O}_\mathbb{C}(V)$, such that there exist $C, M > 0$ satisfying

$$C|z|^M \leq |h_j(z)| \leq \big(C|z|^M\big)^{-1} \qquad (z \in V,\ j = 1, \ldots, r) \ .$$

In particular, since $\Gamma\big(V, \mathscr{S}^t\big(\mathcal{L}^\varphi \otimes \mathcal{R}\big)\big) \simeq \Gamma\big(V, \mathscr{S}\big(\mathcal{L}^\varphi \otimes \mathcal{R}\big)\big) \cap \mathcal{O}^t_{X_{sa}}(V)$ we have that

$$\Gamma\big(V, \mathscr{S}^t\big(\mathcal{L}^\varphi \otimes \mathcal{R}\big)\big) \simeq \begin{cases} \Gamma\big(V, \mathscr{S}\big(\mathcal{L}^\varphi \otimes \mathcal{R}\big)\big) & \text{if } \exp(\varphi) \in \mathcal{O}^t_{X_{sa}}(V) \\ 0 & \text{otherwise} \end{cases} .$$

The conclusion follows by Proposition 2.1.1.

□

We can now state and proof

Proposition 3.1.2 *Let $\varphi_1, \varphi_2 \in z^{-1}\mathbb{C}[z^{-1}]$, $\varphi_2 \neq 0$, $\mathcal{R}_1, \mathcal{R}_2$ regular holonomic $\mathcal{D}_{*0}$-modules. If, for any $\lambda \in \mathbb{R}_{>0}$, $\varphi_1 \neq \lambda\varphi_2$*

$$\mathrm{Hom}_{\mathbb{C}_{X_{sa}}}\big(\mathscr{S}^t\big(\mathcal{L}^{\varphi_1} \otimes \mathcal{R}_1\big), \mathscr{S}^t\big(\mathcal{L}^{\varphi_2} \otimes \mathcal{R}_2\big)\big) \simeq 0 .$$

Otherwise,

$$\mathrm{Hom}_{\mathbb{C}_{X_{sa}}}\big(\mathscr{S}^t\big(\mathcal{L}^{\varphi_1} \otimes \mathcal{R}_1\big), \mathscr{S}^t\big(\mathcal{L}^{\varphi_2} \otimes \mathcal{R}_2\big)\big) \simeq \mathrm{Hom}_{\mathcal{D}_{*0}}\big(\mathcal{R}_1, \mathcal{R}_2\big) \tag{3.1}$$

functorially in $\mathcal{R}_1, \mathcal{R}_2$.

Proof. Suppose that, for any $\lambda \in \mathbb{R}_{>0}$, $\varphi_1 \neq \lambda\varphi_2$. By Proposition 2.2.1*(ii)*, there exists $W \in \mathrm{Op}^c(X_{sa})$ such that

(i) there exists $A_0 > 0$ such that, for any $A \geq A_0$, $W \subset U_{\varphi_1,A}$,

(ii) for any $B > 0$, $W \not\subset U_{\varphi_2,B}$.

In particular, for any $A > A_0$ and $B > 0$,

$$U_{\varphi_1,A} \not\subset U_{\varphi_2,B} . \tag{3.2}$$

Combining (3.2) and the fact that $\mathscr{S}\big(\mathcal{R}_1\big)$ and $\mathscr{S}\big(\mathcal{R}_2\big)$ are locally constant sheaves on $\mathbb{C}^\times$, we obtain, for any $A > A_0$ and $B > 0$,

$$\mathrm{Hom}_{\mathbb{C}_X}\big(\mathscr{S}\big(\mathcal{R}_1\big)_{U_{\varphi_1,A}}, \mathscr{S}\big(\mathcal{R}_2\big)_{U_{\varphi_2,B}}\big) = 0 .$$

Now, using Lemma 3.1.1, we obtain

$$\begin{aligned} &\mathrm{Hom}_{\mathbb{C}_{X_{sa}}}\big(\mathscr{S}^t\big(\mathcal{L}^{\varphi_1} \otimes \mathcal{R}_1\big), \mathscr{S}^t\big(\mathcal{L}^{\varphi_2} \otimes \mathcal{R}_2\big)\big) \simeq \\ &\qquad \simeq \varprojlim_{A>0} \varinjlim_{B>0} \mathrm{Hom}_{\mathbb{C}_X}\big(\mathscr{S}\big(\mathcal{R}_1\big)_{U_{\varphi_1,A}}, \mathscr{S}\big(\mathcal{R}_2\big)_{U_{\varphi_2,B}}\big) \\ &\qquad = 0 . \end{aligned}$$

Suppose now there exists $\lambda \in \mathbb{R}_{>0}$ such that $\varphi_1 = \lambda\varphi_2$. Then

$$U_{\varphi_1,\lambda A} = U_{\varphi_2,A} . \tag{3.3}$$

We need the following

Lemma 3.1.3 *Let $\varphi \in z^{-1}\mathbb{C}[z^{-1}] \setminus \{0\}$ and $A \in \mathbb{R}_{>0}$. The set*

$$U_{\varphi,A} := \{z \in \mathbb{C}^\times;\ \operatorname{Re}\varphi(z) < A\}$$

is homotopically equivalent to $\mathbb{C}^\times$.

Proof. We prove the result in three steps: $\varphi = 1/z$, $\varphi = 1/z^n$ and $\varphi \in z^{-1}\mathbb{C}[z^{-1}]$.

First suppose that $\varphi(z) = \frac{1}{z}$. Then $U_{\varphi,A}$ is the complementary of a closed ball and the result is obvious.

Suppose now that $\varphi(z) = \frac{1}{z^n}$, for some $n \in \mathbb{Z}_{>0}$. Let $\mu_n : \mathbb{C} \to \mathbb{C}$, $z \mapsto z^n$. Then $U_{\varphi,A} = \mu_n^{-1}(U_{\frac{1}{z},A})$ and the conclusion follows.

Suppose now that $\varphi \in z^{-1}\mathbb{C}[z^{-1}]$ and $-v(\varphi) = n$. Mimicking the proof of Proposition 2.1.6, there exists a biholomorphism η such that $\varphi(\eta(z)) = \frac{1}{z^n}$. The conclusion follows. $\square$

Let us conclude the proof of Proposition 3.1.2.

We have the following sequence of isomorphisms

$$\begin{aligned}
&\operatorname{Hom}_{\mathbb{C}_{X_{sa}}}\big(\mathscr{S}^t(\mathcal{L}^{\varphi_1} \otimes \mathcal{R}_1), \mathscr{S}^t(\mathcal{L}^{\varphi_2} \otimes \mathcal{R}_2)\big) \simeq \\
&\qquad \simeq \varprojlim_{A>0} \varinjlim_{B>0} \operatorname{Hom}_{\mathbb{C}_X}\big(\mathscr{S}(\mathcal{R}_1)_{U_{\varphi_1,A}}, \mathscr{S}(\mathcal{R}_2)_{U_{\varphi_2,B}}\big) \\
&\qquad \simeq \varprojlim_{A>0} \varinjlim_{B>0} \operatorname{Hom}_{\mathbb{C}_X}\big(\mathscr{S}(\mathcal{R}_1)_{U_{\varphi_1,A}}, \mathscr{S}(\mathcal{R}_2)_{U_{\varphi_1,A}}\big) \\
&\qquad \simeq \varprojlim_{A>0} \varinjlim_{B>0} \operatorname{Hom}_{\mathbb{C}_X}\big(\mathscr{S}(\mathcal{R}_1)_{\mathbb{C}^\times}, \mathscr{S}(\mathcal{R}_2)_{\mathbb{C}^\times}\big) \\
&\qquad \simeq \operatorname{Hom}_{\mathcal{D}_{*0}}(\mathcal{R}_1, \mathcal{R}_2)\ ,
\end{aligned}$$

where the first isomorphism follows from Lemma 3.1.1, the second from (3.3) and the third from Lemma 3.1.3.

The conclusion follows. $\square$

We can now state the main results of this subsection.

Theorem 3.1.4 *Let $\bigoplus_{\varphi\in\Sigma_1} \mathcal{L}^\varphi \otimes \mathcal{R}_\varphi$ and $\bigoplus_{\psi\in\Sigma_2} \mathcal{L}^\psi \otimes \mathcal{P}_\psi$ be two good models. The following conditions are equivalent.*

(i) $\mathscr{S}^t\Big(\bigoplus_{\varphi\in\Sigma_1} \mathcal{L}^\varphi \otimes \mathcal{R}_\varphi\Big)_{X\setminus\{0\}} \simeq \mathscr{S}^t\Big(\bigoplus_{\psi\in\Sigma_2} \mathcal{L}^\psi \otimes \mathcal{P}_\psi\Big)_{X\setminus\{0\}}$.

(ii) There exist $\varphi_1, \dots, \varphi_d \in z^{-1}\mathbb{C}[z^{-1}]$ such that, $\coprod_{j=1}^{d} \mathbb{R}_{>0}\varphi_j = \mathbb{R}_{>0}\Sigma_1 = \mathbb{R}_{>0}\Sigma_2$ *and, for any $j = 1, \dots, d$,* $\bigoplus_{\varphi\in\Sigma_1\cap\mathbb{R}_{>0}\varphi_j} \mathcal{R}_\varphi \simeq \bigoplus_{\varphi\in\Sigma_2\cap\mathbb{R}_{>0}\varphi_j} \mathcal{P}_\psi$.

Proof. $(ii) \Rightarrow (i)$ Combining Proposition 3.1.2 and the fact that $\mathscr{S}^t(\cdot)_{X\setminus\{0\}}$ is fully faithfull on the category of regular holonomic $\mathcal{D}_{*0}$-modules, we have that

$$\begin{aligned}
\mathrm{Hom}_{\mathbb{C}_{X_{sa}}}\Big(\mathscr{S}^t\Big(\bigoplus_{\varphi\in\Sigma_1}\mathcal{L}^\varphi\otimes\mathcal{R}_\varphi\Big)_{X\setminus\{0\}},\mathscr{S}^t\Big(\bigoplus_{\psi\in\Sigma_2}\mathcal{L}^\psi\otimes\mathcal{P}_\psi\Big)_{X\setminus\{0\}}\Big) &\simeq \\
\simeq \bigoplus_{\varphi\in\Sigma_1}\bigoplus_{\psi\in\Sigma_2}\mathrm{Hom}_{\mathcal{D}_{*0}}\big(\mathscr{S}^t(\mathcal{L}^\varphi\otimes\mathcal{R}_\varphi),\mathscr{S}^t(\mathcal{L}^\psi\otimes\mathcal{P}_\psi)\big) & \\
\simeq \bigoplus_{j=1}^{d}\bigoplus_{\varphi\in\Sigma_1\cap\mathbb{R}_{>0}\varphi_j}\mathrm{Hom}_{\mathcal{D}_{*0}}\big(\mathscr{S}^t(\mathcal{L}^\varphi\otimes\mathcal{R}_\varphi),\bigoplus_{\psi\in\Sigma_2\cap\mathbb{R}_{>0}\varphi_j}\mathscr{S}^t(\mathcal{L}^\psi\otimes\mathcal{P}_\psi)\big) & \\
\simeq \bigoplus_{j=1}^{d}\mathrm{Hom}_{\mathcal{D}_{*0}}\big(\bigoplus_{\varphi\in\Sigma_1\cap\mathbb{R}_{>0}\varphi_j}\mathcal{R}_\varphi,\bigoplus_{\psi\in\Sigma_2\cap\mathbb{R}_{>0}\varphi_j}\mathcal{P}_\psi\big)\ . &
\end{aligned}$$

The functoriality of (3.1) allows to conclude.

$(i) \Rightarrow (ii)$ First let us suppose that $\mathbb{R}_{>0}\Sigma_1 \neq \mathbb{R}_{>0}\Sigma_2$. Hence either $\mathbb{R}_{>0}\Sigma_1 \not\subset \mathbb{R}_{>0}\Sigma_2$ or $\mathbb{R}_{>0}\Sigma_2 \not\subset \mathbb{R}_{>0}\Sigma_1$. Suppose the latter.

There exists $\psi \in \Sigma_2$ such that for any $\varphi \in \Sigma_1, \lambda \in \mathbb{R}_{>0}$, $\psi \neq \lambda\varphi$.

Suppose that $\psi \neq 0$. By Proposition 3.1.2, we have

$$\mathrm{Hom}_{\mathbb{C}_{X_{sa}}}\big(\mathscr{S}^t\big(\bigoplus_{\varphi\in\Sigma_1}\mathcal{L}^\varphi\otimes\mathcal{R}_\varphi\big)_{X\setminus\{0\}},\mathscr{S}^t\big(\mathcal{L}^\psi\otimes\mathcal{P}_\psi\big)_{X\setminus\{0\}}\big)\simeq 0\ .$$

It follows that

$$\mathscr{S}^t\big(\bigoplus_{\varphi\in\Sigma_1}\mathcal{L}^\varphi\otimes\mathcal{R}_\varphi\big)_{X\setminus\{0\}}\not\simeq\mathscr{S}^t\big(\bigoplus_{\psi\in\Sigma_2}\mathcal{L}^\psi\otimes\mathcal{P}_\psi\big)_{X\setminus\{0\}}\ .$$

Suppose that $\psi = 0$, then $0 \notin \Sigma_1$ and, by Proposition 3.1.2,

$$\mathrm{Hom}_{\mathbb{C}_{X_{sa}}}\big(\mathscr{S}^t\big(\mathcal{L}^\psi\otimes\mathcal{P}_\psi\big)_{X\setminus\{0\}},\mathscr{S}^t\big(\bigoplus_{\varphi\in\Sigma_1}\mathcal{L}^\varphi\otimes\mathcal{R}_\varphi\big)_{X\setminus\{0\}}\big)\simeq 0\ .$$

It follows that

$$\mathscr{S}^t\big(\bigoplus_{\varphi\in\Sigma_1}\mathcal{L}^\varphi\otimes\mathcal{R}_\varphi\big)_{X\setminus\{0\}}\not\simeq\mathscr{S}^t\big(\bigoplus_{\psi\in\Sigma_2}\mathcal{L}^\psi\otimes\mathcal{P}_\psi\big)_{X\setminus\{0\}}\ .$$

The case $\mathbb{R}_{>0}\Sigma_1 \not\subset \mathbb{R}_{>0}\Sigma_2$ is treated similarly.

Now let us suppose that $\mathbb{R}_{>0}\Sigma_1 = \mathbb{R}_{>0}\Sigma_2 = \coprod_{j=1}^{d}\mathbb{R}_{>0}\varphi_j$ and, there exists $j' \in \{1,\dots,d\}$, such that

$$\bigoplus_{\varphi\in\Sigma_1\cap\mathbb{R}_{>0}\varphi_{j'}}\mathcal{R}_\varphi\not\simeq\bigoplus_{\psi\in\Sigma_2\cap\mathbb{R}_{>0}\varphi_{j'}}\mathcal{P}_\psi\ .$$

By Proposition 3.1.2 we have that

$$\begin{aligned}
\mathrm{Hom}_{\mathbb{C}_{X_{sa}}}\big(\mathscr{S}^t\Big(\bigoplus_{\varphi\in\Sigma_1}\mathcal{L}^\varphi\otimes\mathcal{R}_\varphi\Big)_{X\setminus\{0\}},\mathscr{S}^t\Big(\bigoplus_{\psi\in\Sigma_2}\mathcal{L}^\psi\otimes\mathcal{P}_\psi\Big)_{X\setminus\{0\}}\big) &\simeq \\
\simeq \bigoplus_{j=1}^{d}\mathrm{Hom}_{\mathcal{D}_{*0}}\big(\bigoplus_{\varphi\in\Sigma_1\cap\mathbb{R}_{>0}\varphi_j}\mathcal{R}_\varphi,\bigoplus_{\psi\in\Sigma_2\cap\mathbb{R}_{>0}\varphi_j}\mathcal{P}_\psi\big)\ . &
\end{aligned}$$

The functoriality of (3.1) allows to conclude.

□

We conclude the study of tempered solutions of good models with

Theorem 3.1.5 *Let $\omega \in z^{-1}\mathbb{C}[z^{-1}]$, $-v(\omega) \geq k$. The functor*

$$\begin{array}{rccl} \mathscr{S}^t_\omega(\cdot): & \mathsf{GM}_k & \longrightarrow & \mathrm{Mod}(\mathbb{C}_{X_{sa}}) \\ & \mathcal{M} & \longmapsto & \mathcal{H}om_{\varrho_!\mathcal{D}_X}(\varrho_!(\mathcal{M}\otimes\mathcal{L}^\omega), \mathcal{O}^t_{X_{sa}}) \ , \end{array}$$

is fully faithful.

Proof. Clearly it is sufficient to prove that, given $\varphi_1, \varphi_2 \in z^{-1}\mathbb{C}[z^{-1}]$, $k > \max\{-v(\varphi_1), -v(\varphi_2)\}$, $\mathcal{R}_1, \mathcal{R}_2$ regular holonomic $\mathcal{D}_{*0}$-modules, the functor of tempered solutions induces the isomorphism

(3.4)

$$\mathrm{Hom}_{\mathcal{D}_{*0}}\big(\mathcal{L}^{\varphi_1}\otimes\mathcal{R}_1, \mathcal{L}^{\varphi_2}\otimes\mathcal{R}_2\big) \simeq \mathrm{Hom}_{\mathbb{C}_{X_{sa}}}\big(\mathscr{S}^t_\omega\big(\mathcal{L}^{\varphi_1}\otimes\mathcal{R}_1\big), \mathscr{S}^t_\omega\big(\mathcal{L}^{\varphi_2}\otimes\mathcal{R}_2\big)\big) \ .$$

Let us prove (3.4).

First, suppose that $\varphi_1 \neq \varphi_2$. Then

$$\mathrm{Hom}_{\mathcal{D}_{*0}}\big(\mathcal{L}^{\varphi_1}\otimes\mathcal{R}_1, \mathcal{L}^{\varphi_2}\otimes\mathcal{R}_2\big) = 0 \ .$$

Moreover, as, for any $\lambda \in \mathbb{R}_{>0}$, $\lambda(\varphi_1+\omega) \neq \varphi_2+\omega$, Proposition 3.1.2 implies that,

$$\mathrm{Hom}_{\mathbb{C}_{X_{sa}}}\big(\mathscr{S}^t_\omega\big(\mathcal{L}^{\varphi_1}\otimes\mathcal{R}_1\big), \mathscr{S}^t_\omega\big(\mathcal{L}^{\varphi_2}\otimes\mathcal{R}_2\big)\big) = 0 \ .$$

Now, suppose that $\varphi_1 = \varphi_2$. The result follows from Proposition 3.1.2 and the fact that

$$\mathrm{Hom}_{\mathcal{D}_{*0}}\big(\mathcal{R}_1, \mathcal{R}_2\big) \simeq \mathrm{Hom}_{\mathcal{D}_{*0}}\big(\mathcal{L}^{\varphi_1}\otimes\mathcal{R}_1, \mathcal{L}^{\varphi_2}\otimes\mathcal{R}_2\big) \ .$$

□

3.2 Tempered solutions of ordinary differential equations

We begin this subsection by proving the analogue of Lemma 3.1.1 in the case of ramified determinant polynomials i.e. with non-integer exponents.

Recall that, for $Y \subset X$, $Y_{X_{sa}}$ is the subanalytic site on Y induced by X_{sa}, in particular the open sets of $Y_{X_{sa}}$ are of the form $U \cap Y$ for $U \in \mathrm{Op}^c(X_{sa})$. For $\mathcal{F} \in \mathrm{Mod}(k_{X_{sa}})$, we denote by $\mathcal{F}|_Y$ the restriction of $\mathcal{F}$ to $Y_{X_{sa}}$.

Recall the definitions of $\Omega(\mathcal{M})$ and $r_{\varphi,\mathcal{M}}$ (resp. $U_{\varphi,\epsilon}$) given in Definition 1.3.6 (resp. Corollary 2.1.3).

Lemma 3.2.1 *Let $\mathcal{M} \in \mathrm{Mod}_h(\mathcal{D}_{*0})$, $\vartheta \in \mathbb{R}$, $Y := X \setminus (\mathbb{R}_{\geq 0} e^{i\vartheta})$. Then*

$$\mathscr{S}^t\big(\mathcal{M}\big)\big|_Y \simeq \bigoplus_{\varphi \in \Omega(\mathcal{M})} \varinjlim_{\epsilon > 0} \varrho_* \, \mathbb{C}^{r_{\varphi,\mathcal{M}}}_{Y, U_{\varphi,\epsilon}} \; .$$

Proof. As $\mathcal{M}$ is fixed, for sake of simplicity, we drop the index $\mathcal{M}$ in the symbol $r_{\varphi,\mathcal{M}}$.

Let $V \in \mathrm{Op}(Y_{X_{sa}})$ connected. By the Hukuhara-Turrittin's Asymptotic Theorem 1.3.3, the $\mathbb{C}$-vector space $\mathscr{S}\big(\mathcal{M}\big)(V) \subset \mathcal{O}(V)$ is generated by $\big\{h_{\varphi,j} \exp(\varphi)\big\}_{\substack{\varphi \in \Omega(\mathcal{M}) \\ j \in \{1,\ldots,r_\varphi\}}}$.

Hence

$$\begin{aligned} \mathscr{S}^t\big(\mathcal{M}\big)(V) &\simeq \mathscr{S}\big(\mathcal{M}\big)(V) \cap \mathcal{O}^t(V) \\ &\simeq \Big\{ \sum_{\varphi \in \Omega(\mathcal{M})} \sum_{j=1}^{r_\varphi} c_{\varphi,j} h_{\varphi,j} \exp(\varphi) \in \mathcal{O}^t(V);\; c_{\varphi,j} \in \mathbb{C} \Big\} \, . \end{aligned}$$

Since, for $\varphi \in \Omega(\mathcal{M}), j \in \{1, \ldots, r_\varphi\}$, $h_{\varphi,j} \exp(\varphi)$ are $\mathbb{C}$-linearly independent functions and $h_{\varphi,j}, h^{-1}_{\varphi,j} \in \mathcal{O}^t(V)$, one has that

$$\sum_{\varphi \in \Omega(\mathcal{M})} \sum_{j=1}^{r_\varphi} c_{\varphi,j} h_{\varphi,j} \exp(\varphi) \in \mathcal{O}^t(V)$$

if and only if $\exp(\varphi) \in \mathcal{O}^t(V)$ for $c_{\varphi,j} \neq 0$.

The conclusion follows. □

We denote by for the functor from the category of Ω-filtered local systems to the category of local systems on S^1.

Theorem 3.2.2 *Let $k \in \mathbb{Z}_{>0}$, $\mathcal{M}_1, \mathcal{M}_2 \in \mathrm{Mod}_h(\mathcal{D}_{*0})_k$ and $\omega \in z^{-1}\mathbb{C}[z^{-1}]$ such that $-v(\omega) > k$. The following conditions are equivalent.*

(i) $\mathscr{S}^t_\omega\big(\mathcal{M}_1\big)_{X\setminus\{0\}} \simeq \mathscr{S}^t_\omega\big(\mathcal{M}_2\big)_{X\setminus\{0\}}$.

(ii) *(a)* $\mathsf{for}\big(\mathscr{S}^\Omega(\mathcal{M}_1)\big) \simeq \mathsf{for}\big(\mathscr{S}^\Omega(\mathcal{M}_2)\big)$ *and*

(b) for any $\vartheta \in S^1$, $\mathscr{S}^\Omega\big(\mathcal{M}_1\big)_\vartheta \simeq \mathscr{S}^\Omega\big(\mathcal{M}_2\big)_\vartheta$ as Ω_ϑ-graded $\mathbb{C}$-vector spaces.

Proof. *(i)⇒(ii).* First remark that $\mathsf{for}\big(\mathscr{S}^\Omega(\mathcal{M}_1)\big) \simeq \mathsf{for}\big(\mathscr{S}^\Omega(\mathcal{M}_2)\big)$ if and only if $\mathscr{S}\big(\mathcal{M}_1\big)_{X\setminus\{0\}} \simeq \mathscr{S}\big(\mathcal{M}_2\big)_{X\setminus\{0\}}$. Since $\varrho^{-1}\mathscr{S}^t\big(\mathcal{M}_j\big)_{X\setminus\{0\}} \simeq \mathscr{S}\big(\mathcal{M}_j\big)_{X\setminus\{0\}}$, the condition *(a)* is proved.

Suppose now that there exists $\vartheta \in S^1$ such that $\mathscr{S}^\Omega(\mathcal{M}_1)_\vartheta \not\simeq \mathscr{S}^\Omega(\mathcal{M}_2)_\vartheta$. Then, either $\Omega(\mathcal{M}_1) \neq \Omega(\mathcal{M}_2)$ or there exists $\varphi \in \Omega(\mathcal{M}_1) \cap \Omega(\mathcal{M}_2)$ such

that $r_{\varphi,\mathcal{M}_1} \neq r_{\varphi,\mathcal{M}_2}$. In the former case, combining the ideas of the first part of the proof of Proposition 3.1.2 with Lemma 3.2.1 and Corollaries 2.1.3 and 2.2.2, we obtain that, for any $\vartheta \in \mathbb{R}$, $\mathscr{S}^t_\omega(\mathcal{M}_1)|_{X\setminus\mathbb{R}_{\geq 0}e^{i\vartheta}} \not\simeq \mathscr{S}^t_\omega(\mathcal{M}_2)|_{X\setminus\mathbb{R}_{\geq 0}e^{i\vartheta}}$. In the latter case the result follows easily from Lemma 3.2.1.

(ii)⇒(i). Set $\mathscr{S}_\omega(\cdot) := \mathscr{S}(\cdot \otimes \mathcal{L}^\omega)$ $(j = 1, 2)$.

Let $\vartheta_1, \vartheta_2 \in \mathbb{R}$, $\vartheta_1 \neq \vartheta_2 \pmod{2\pi}$, $Y_j := X \setminus \mathbb{R}_{\geq 0}e^{i\vartheta_j}$ $(j = 1, 2)$.

Since for any $\vartheta \in S^1$, $\mathscr{S}^\Omega\big(\mathcal{M}_1\big)_\vartheta \simeq \mathscr{S}^\Omega\big(\mathcal{M}_2\big)_\vartheta$, then $\Omega(\mathcal{M}_1) = \Omega(\mathcal{M}_2)$ and $r_{\varphi,\mathcal{M}_1} = r_{\varphi,\mathcal{M}_2}$. In particular, Lemma 3.2.1 implies that

$$\mathscr{S}^t_\omega(\mathcal{M}_1)|_{Y_1} \simeq \mathscr{S}^t_\omega(\mathcal{M}_2)|_{Y_1}$$

$$\mathscr{S}^t_\omega(\mathcal{M}_1)|_{Y_2} \simeq \mathscr{S}^t_\omega(\mathcal{M}_2)|_{Y_2}\ .$$

Now, since $\mathsf{for}\big(\mathscr{S}^\Omega(\mathcal{M}_1)\big) \simeq \mathsf{for}\big(\mathscr{S}^\Omega(\mathcal{M}_2)\big)$, we have that $\mathscr{S}\big(\mathcal{M}_1\big)_{X\setminus\{0\}} \simeq \mathscr{S}\big(\mathcal{M}_2\big)_{X\setminus\{0\}}$ which implies $\mathscr{S}_\omega\big(\mathcal{M}_1\big)_{X\setminus\{0\}} \simeq \mathscr{S}_\omega\big(\mathcal{M}_2\big)_{X\setminus\{0\}}$. We conclude thanks to the commutative diagram below. Roughly speaking, it says that $\mathscr{S}^t_\omega(\mathcal{M}_j)_{X\setminus\{0\}}$ is completely determined by $\mathscr{S}^t_\omega(\mathcal{M}_j)|_{Y_1}$, $\mathscr{S}^t_\omega(\mathcal{M}_j)|_{Y_2}$ and $\mathscr{S}_\omega(\mathcal{M}_j)_{X\setminus\{0\}}$ $(j = 1, 2)$.

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{S}^t_\omega(\mathcal{M}_j)_{X\setminus\{0\}} \arrow[r] \arrow[d, hook] & \mathscr{S}^t_\omega(\mathcal{M}_j)_{Y_1} \oplus \mathscr{S}^t_\omega(\mathcal{M}_j)_{Y_2} \arrow[r] \arrow[d, hook] & \mathscr{S}^t_\omega(\mathcal{M}_j)_{Y_1\cap Y_2} \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \mathscr{S}_\omega(\mathcal{M}_j)_{X\setminus\{0\}} \arrow[r] & \mathscr{S}_\omega(\mathcal{M}_j)_{Y_1} \oplus \mathscr{S}_\omega(\mathcal{M}_j)_{Y_2} \arrow[r] & \mathscr{S}_\omega(\mathcal{M}_j)_{Y_1\cap Y_2} \arrow[r] & 0
\end{tikzcd}$$

□

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